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Transmuted Lindley Distribution

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Abstract

In this article, we generalize the Lindley distribution using the quadratic rank transmutation map studied by Shaw et al. [8] to develop a transmuted Lindley distribution. We provide a comprehensive description of the mathematical properties of the subject distribution along with its reliability behavior. The usefulness of the transmuted Lindley distribution for modeling reliability data is illustrated using real data.

Keywords: *Lindley Distribution, hazard rate function, reliability function, parameter estimation.*

2010 Mathematics Subject Classification: 62N05, 90B25.

1 Introduction

In many applied sciences such as medicine, engineering and finance, amongst others, modeling and analyzing lifetime data are crucial. Several lifetime distributions have been used to model such kinds of data. The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with revelent statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

In this article we present a new generalization of Lindley distribution called the transmuted Lindley distribution.

Definition 1.1 *A random variable X is said to have transmuted distribution if its cumulative distribution function(cdf) is given by*

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), |\lambda| \leq 1. \tag{1}$$

where $G(x)$ is the cdf of the base distribution.

Observe that at $\lambda = 0$ we have the distribution of the base random variable. Aryal et al. [3] studied the the transmuted Gumbel distribution and it has been observed that transmuted Gumbel distribution can be used to model climate data. In the present study we will provide mathematical formulation of the transmuted Lindley distribution and some of its properties.

2 Transmuted Lindley Distribution

Definition 2.1 *A random variable X is said to have the Lindley distribution with parameter θ if its probability density is defined as*

$$f(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x > 0, \theta > 0. \tag{2}$$

The corresponding cumulative distribution function (c.d.f.) is:

$$F(x) = 1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}, \quad x > 0, \theta > 0 \tag{3}$$

Now using (1) and (3) we have the cdf of a transmuted Lindley distribution

$$F_1(x) = \left(1 - \frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\left(1 + \lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right) \tag{4}$$

Hence, the pdf of transmuted Lindley distribution with parameter λ is

$$f_1(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left(1 - \lambda + 2\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right) \tag{5}$$

Note that the transmuted Lindley distribution is an extended model to analyze more complex data and it generalizes some of the widely used distributios.The Lindley distribution is clearly a special case for $\lambda = 0$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted Lindley distribution for selected values of the parameters λ and θ. Figure 2 illustrates some of the possible shapes of the CDF of a transmuted Lindley distribution for selected values of the parameter λ by keeping $\theta = 1$.

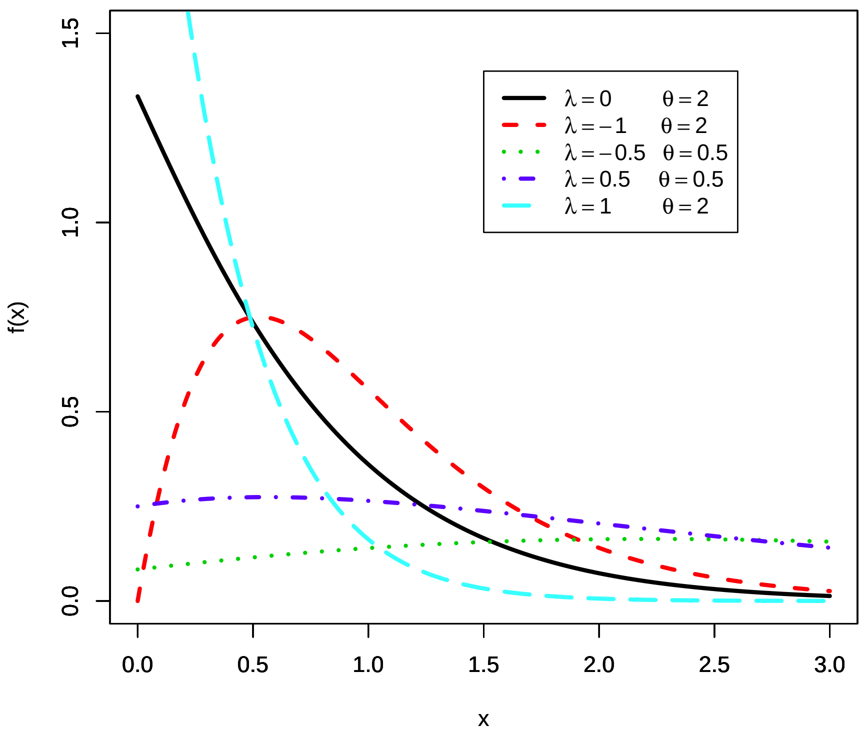

Figure 1: Pdf of Transmuted Lindley distribution

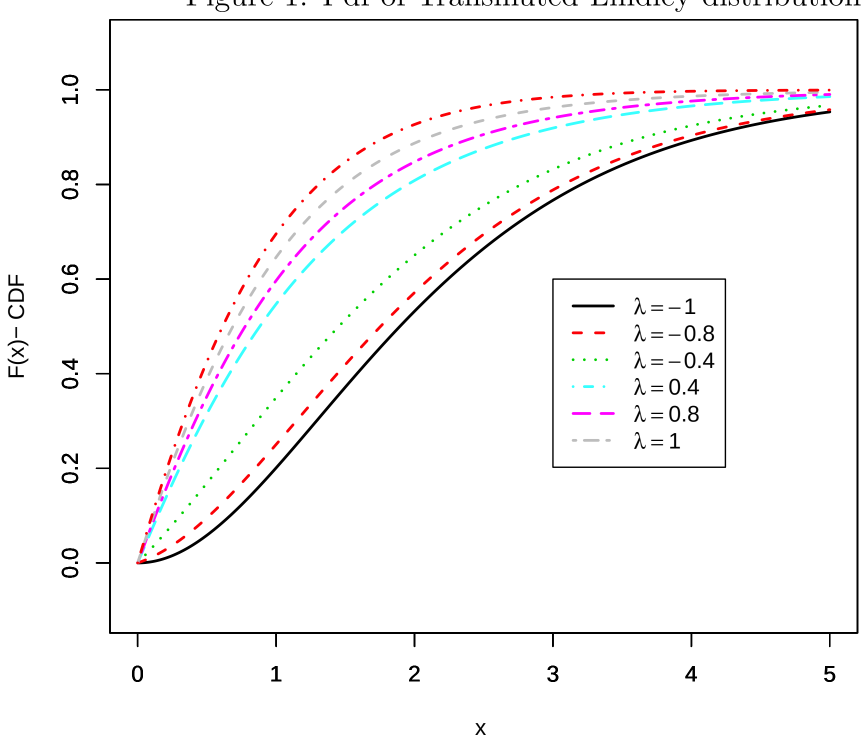

Figure 2: CDF of Transmuted Lindley distribution

3 Moments

Now let us consider the different moments of the transmuted Lindley distribution. Suppose X denote the transmuted Lindley distribution random variable with parameter $\theta$and λ, then

$$\begin{aligned} E(X^k) &= \frac{\theta^2}{\theta+1}\int_0^\infty x^k(1+x)e^{-\theta x}\Big(1-\lambda+2\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big) \\ &= \frac{k!}{\theta^k(\theta+1)}\big[(1-\lambda)(\theta+k+1)+\frac{\lambda\cdot\theta}{2^{k-1}(\theta+1)}(2\theta+3\theta+k)\big] \end{aligned}$$

Here we use $\int_0^\infty x^k e^{-\theta x}dx = k!\theta^{-k-1}$ (see [9], 2.3.3.2 page 322) Therefore putting $k=1$, we obtain the mean as

$$E(X) = \frac{1}{\theta(\theta+1)}\big[(1-\lambda)(\theta+2)+\frac{2\lambda\cdot\theta}{(\theta+1)}(\theta+2)\big] \tag{6}$$

and putting $k=2$ we obtain the second moment as

$$E(X^2) = \frac{2}{\theta^2(\theta+1)}\big[(1-\lambda)(\theta+3)+\frac{\lambda\cdot\theta}{2(\theta+1)}(2\theta+5)\big] \tag{7}$$

The maximum likelihood estimates, MLE's, of the parameters that are inherent within the transmuted exponentiated Lindley probability distribution function is given by the following:

$$L = \frac{\theta^{2n}}{(\theta+1)^n}e^{-\sum\limits_{i=1}^{n}\theta x_i}\prod_{i=1}^{n}(1+x_i)\cdot\Big(1-\lambda+2\lambda\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}\Big) \tag{8}$$

$$\begin{aligned} \ln L &= 2n\ln\theta - n\ln(\theta+1)-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\ln(1+x_i) \\ &+\sum_{i=1}^{n}\ln(1-\lambda+2\lambda\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}) \end{aligned}$$

Now setting

$$\frac{\partial\ln L}{\partial\theta}=0 \quad and \quad \frac{\partial\ln L}{\partial\lambda}=0,$$

we have

$$0=\frac{2n}{\theta}-\frac{n}{\theta+1}-\sum_{i=1}^{k}x_i+\sum_{i=1}^{k}\frac{2\lambda x_i e^{-\theta x_i}\left[\frac{1}{(\theta+1)^2}-\frac{\theta+1+\theta x_i}{\theta+1}\right]}{1-\lambda+2\lambda\frac{\theta+1+\theta x_i}{\theta+1}e^{-\theta x_i}}$$

$$0 = \sum_{i=1}^{n} \frac{2\frac{\theta+1+\theta x_i}{\theta+1} e^{-\theta x_i} - 1}{1 - \lambda + 2\lambda \frac{\theta+1+\theta x_i}{\theta+1} e^{-\theta x_i}}$$

The maximum likelihood estimator $\hat{\beta} = (\hat{\theta}, \hat{\lambda})'$ of $\beta = (\theta, \lambda)'$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function given in (8).In order to compute the standard error and asymptotic confidence interval we use the usual large sample approximation in which the maximum likelihood estimators of θ can be treated as being approximately trivariate normal. Hence as $n \to \infty$ the asymptotic distribution of the MLE $(\hat{\theta}, \hat{\lambda})$ is given by, see Zaindin et al. [10],

$$\begin{pmatrix} \hat{\theta} \\ \hat{\lambda} \end{pmatrix} \sim N \left[\begin{pmatrix} \theta \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V_{11}} & \hat{V_{12}} \\ \hat{V_{21}} & \hat{V_{22}} \end{pmatrix} \right]$$

where, $V_{ij} = V_{ij|\theta=\hat{\theta}}$ and

$$\begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1}$$

is the approximate variance covariance matrix with its elements obtained from

$$A_{11} = -\frac{\partial^2 \ln L}{\partial \alpha^2}, \qquad A_{12} = -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta},$$

$$A_{22} = -\frac{\partial^2 \ln L}{\partial \beta^2}.$$

where $\ln L$ is the log-likelihood function given in (**??**). An approximate $100(1-\alpha)\%$ two sided confidence intervals for α, β and λ are, respectively, given by

$$\hat{\theta} \pm z_{\alpha/2}\sqrt{\hat{V_{11}}}, \text{ and } \hat{\lambda} \pm z_{\alpha/2}\sqrt{\hat{V_{22}}}$$

where z_α is the upper $\alpha-$th percentiles of the standard normal distribution. Using R we can easily compute the Hessian matrix and its inverse and hence the values of the standard error and asymptotic confidence intervals.

4 Reliability Analysis

The reliability function $R(x)$, which is the probability of an item not failing prior to some time t, is defined by $R(x) = 1 - F(x)$. The reliability function of a transmuted Lindley distribution is given by

$$R(x) = \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x} \left(1 - \lambda + \lambda \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}\right) \tag{9}$$

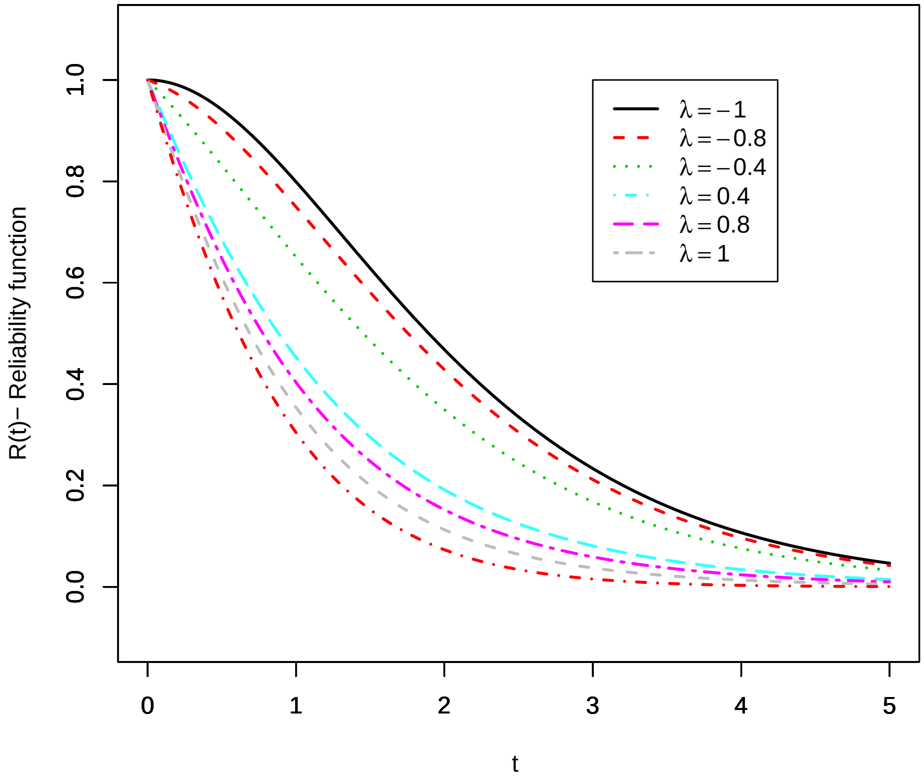

Figure 3: The reliability function of a transmuted Lindley distribution

The other characteristic of interest of a random variable is the hazard rate function defined by

$$h(x) = \frac{f(x)}{1 - F(x)}$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to the time t. The hazard rate function for a transmuted Lindley random variable is given by

$$h(x) = \frac{\theta^2(1+x)}{\theta+1+\theta x} \frac{1 - \lambda + 2\lambda \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}}{\lambda - 1 - \lambda \frac{\theta+1+\theta x}{\theta+1} e^{-\theta x}} \tag{10}$$

Figure 2 illustrates the reliability behavior behavior of a transmuted Lindley distribution as the value of the parameter λ varies from -1 to 1, keeping $\theta = 1$.

The hazard rate function $h(x)$ of a transmuted Lindley distribution has the following properties:

- if $\lambda = 1$, the failure rate is decreasing .
- $h(x)$ tends to $-\frac{\theta^2}{\theta+1}(1+\lambda)$ and $-\theta$ where $x \to 0$ and $x \to \infty$, respectively.

Proof. For $\lambda = 1$, we have

$$h(x) = -2\frac{\theta^2(1+x)}{\theta+1+\theta x}$$

The derivation of $h(x)$ with respect to x is given by

$$\frac{\partial h(x)}{\partial x} = -2\Big(\frac{\theta}{\theta+1+\theta x}\Big)^2 < 0$$

so $h(x)$ is decreassing.

5 Order Statistics

In statistics, the kth order statistic of a statistical sample is equal to its kth-smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max\{X_1, X_2, \ldots, X_n\}.$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$Range\{X_1, X_2, \ldots, X_n\} = X_{(n)} - X_{(1)}.$$

We know that if $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j} \tag{11}$$

for $j = 1, 2, \ldots, n$. The pdf of the j^{th} order statistic for transmuted Lindley distribution is given by

$$\begin{aligned} f_{X_{(j)}}(x) = {} & \frac{n!}{(j-1)!(n-j)!}\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\Big(1-\lambda+2\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big) \\ & \times \Big(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{j-1}\Big(1+\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{j-1} \\ & \times \Big(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{n-j}\Big(1-\lambda+\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{n-j} \end{aligned} \tag{12}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) &= \frac{n\theta^2}{\theta+1}(1+x)e^{-\theta x}\Big(1-\lambda+2\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big) \\ &\times\Big(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{n-1}\Big(1+\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{n-1} \end{aligned}$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$\begin{aligned} f_{X_{(1)}}(x) &= \frac{n\cdot\theta^2}{\theta+1}(1+x)e^{-\theta x}\Big(1-\lambda+2\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big) \\ &\times\Big(\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{n-1}\Big(1-\lambda+\lambda\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\Big)^{n-1} \end{aligned}$$

6 Application

In this section, we use a real data set to show that the transmuted Lindley distribution can be a better model than one based on the Lindley distribution. The data set given in Table 1 represents an uncensored data set corresponding to remission times (in months) of a random sample of 128 bladder cancer patients reported in [12].

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
3.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
9.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
25.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
2.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
7.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
15.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
1.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
7.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
1.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85
8.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02
3.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
21.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49		

Table 1.

The result are as follows:

Model	Parameter Estimates	LL
Exponential	$\hat{\alpha}$=0.097	-426.76
Lindley	$\hat{\theta}$=0.196	-419.52
Transmuted Lindley	$\hat{\theta}=0.156; \hat{\beta}=0.617$	-415.15

Table 2.

Model	-2LL	AIC	AICC	BIC
Exponential	853.52	855.53	855.5617	858.37
Lindley.	839.04	841.06	841.0917	843.91
Trans. Lindley	830.3	834.31	834.406	840.01

Table 3.

In order to compare the distributions, we consider $-2\log(L)$,AIC (Akaike Information Criterion), AICC(Akaike Information Criterion Corrected) and BIC(Bayesian information criterion)for the real data set. The best distribution correspond to lower $-2\log(L)$,AIC, AICC and BIC values.

Table 2 shows parameter MLEs to each one of the three fitted distributions, Table 3 shows the values of $-2\log(L)$,AIC, AICC and BIC values. The values in Table 3, indicate that the transmuted Lindley distribution is a strong competitor to other distributions used here for fitting data.

7 Open Problem

The probability density function of the Lindley Geometric (LG) distribution function is given by(see [11])

$$f(y) = \frac{\theta^2}{\theta+1}(1-p)(1+y)e^{-\theta y}\left[1-p(1+\frac{\theta y}{\theta+1})e^{-\theta y}\right]^{-2} \tag{13}$$

where variable $y > 0$, and parameters $\theta > 0$ and $0 < p < 1$.

Is it transmuted Lindley geometric distribution better than transmuted Lindley distribution for fitting data?

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