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Common Fixed Point Theorems for Sequences of Mappings in Fuzzy Metric Spaces

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Abstract

The purpose of this paper is to study common fixed point theorems for six (four single-valued and two set-valued) mappings in fuzzy metric spaces. without assuming compatibility and continuity of any mapping on non complete metric spaces. To prove the theorem, we use a non compatible condition, that is, weak commutativity of type (Kh)in fuzzy metric spaces. We show that completeness of the whole space is not necessary for the existence and uniqueness of common fixed point. Also, we prove a common fixed point theorem for two self mappings and two sequences set-valued mappings by the same weaker conditions. Our results generalize, extend and improve the corresponding results given by many authors.

Keywords: *Fuzzy metric, Common fixed point, single-valued and set-valued mappings, weakly commuting of type (Kh) in fuzzy metric space.*

1 Introduction

After introduction of fuzzy sets by Zadeh[11], many researchers have defined fuzzy metric spaces in different ways such as Kramosil and Michalek[10]. The concept of compatible mappings has been investigated initially by Jungck [2], by which the notions of commuting and weakly commuting mappings are generalized. In the last years, the concepts of δ-compatible and weakly compatible mappings were introduced by Jungck and Rhoades [3]. In the last few decades, the common fixed point theorems for compatible mappings have applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other applied mathematics[4,6]. Note that common fixed point theorems for single and set-valued maps are interesting and ply a major role in many areas. Abu-Donia, Abd-Rabou [7-8] studied common fixed point theorems for single and set-valued mappings in fuzzy metric spaces.Abd-Rabou [9] studied common fixed point theorems for weakly compatible hybrid mappings. The purpose of this paper is to establish a common fixed point for six mappings under weaker condition, that is, weakly commuting of type (Kh) in fuzzy metric spaces. our results generalize, extend and improve the corresponding results given by many authors.

2 Basic Preliminaries

In this section, we recall some notions and definitions in fuzzy metric.
Definition 2.1[1] A mapping $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a * 1 = a$ for every $a \in [0,1]$

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 2.2 [10] A triplet $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t norm and M is a fuzzy set on $X \times X \times [0,\infty) \rightarrow [0,1]$ satisfying, $\forall x, y \in X$,the following conditions:

(1) $M(x, y, 0) = 0$,

(2) $M(x, y, t) = 1, \forall t > 0$ iff $x = y$,

(3) $M(x, y, t) = M(y, x, t)$

(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, s + t), s, t \in [0, 1)$,

(5) $M(x, y, .) : [0,\infty) \rightarrow [0,1]$ is left continuous.

Note that $M(y, x, t)$ can be thought of as the degree of nearness between x and y with respect to t.
Definition 2.3 [12] A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to a point $x \in X$ if $lim_{n\to\infty} M(x_n, x, t) = 1, \forall t > 0$.
A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is Cauchy sequence if $lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1, \forall t, p > 0$.

A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.4 [**3**] The mappings $I : X \to X$ and $F : X \to B(X)$(The class nonempty bounded subsets of X) are weakly compatible if they commute at coincidence points. i.e. for each point $u \in X$ such that $Iu \in Fu$, we have $FIu = IFu$. Not that the equation $Fu = \{Iu\}$ implies that Fu is singleton.

Definition 2.5[**7**] The mappings $I : X \to X$ and $F : X \to B(X)$ are compatible if, for all $t > 0$, $lim_{n\to\infty} M(FIx_n, IFx_n, t) = 1$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Ix_n = z \in A = lim_{n\to\infty} Fx_n, A \subseteq X$.

Definition 2.6 The mappings $I : X \to X$ and $F : X \to B(X)$ are $R-$ weakly commuting if, for all $R, t > 0$, $M(FIx, IFx, t) \geq M(Fx, Ix, t/R)$, such that $x \in X, IFx \in B(X)$.

Definition 2.7 The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be weakly commuting of type (Kh) at x if, for all $R, t > 0, x \in X$, $M(IIx, FIx, t) \geq M(Fx, Ix, t/R)$.

Here I and F are weakly commuting of type (Kh) on X if the above inequality hold for all $x \in X$.

Remark 2.1 Every weakly compatible pair of hybrid maps is weakly commuting of type (Kh) but the converse is not necessarily true.

In the following example, we know that every metric induces a fuzzy metric

Example 2.1 Let (X, δ) be a metric space. Define $a * b = ab, a \in A, b \in B$ and for all $A, B \subset X, t > 0$,

$$M(x, y, t) = \tfrac{t}{t+\delta(A,B)}$$

We call M is a fuzzy metric on X induced by metric δ.

Example 2.2 Let $X = [1, 10]$. Define $I : X \to X$ and $F : X \to B(X)$ by

$$Ix = \begin{cases} x, & \text{if } 1 \leq x \leq 5 \\ \frac{x+3}{4}, & \text{if } 5 < x \leq 10 \end{cases}, \qquad F(x) = \begin{cases} [1, x], & \text{if } 1 \leq x \leq 2 \\ [2, x], & \text{if } 2 < x \leq 5 \\ [2, \frac{x+1}{3}], & \text{if } 5 < x \leq 10 \end{cases},$$

$\delta(A, B) = max\{d(a, b) : a \in A, b \in B\}, A, B \in B(X)$, where $d(a, b) = |a - b|$. Let $x_n = 5 + \frac{1}{n}, n = 1, 2, ...$ Then, $\lim\limits_{n\longrightarrow\infty} Ix_n = 2$ and $\lim\limits_{n\longrightarrow\infty} Fx_n = \{2\}$. Also $IFx_n \in B(X)$ and $M(FIx_n, IFx_n, t) = M([2, 2 + \frac{1}{4n}], [2, 2 + \frac{1}{3n}], t) \to 1$, as $n \to \infty$.

Hence, I and F are δ-compatible and hence weakly compatible. On the other hand if we take $x = 2$, then $IIx = 2$, $FIx = [1, 2]$ and clearly I and F are weakly commuting of type (Kh) for $x = 2$.

Example 2.3 Let $X = [1, \infty)$. Define $I : X \to X$ and $F : X \to B(X)$ by $Ix = 2x$ and $Fx = [1, x]$ for all $x \in X$, $\delta(A, B) = max\{d(a, b) : a \in A, b \in B\}, A, B \in B(x)$, where $d(a, b) = |a - b|$. Then $IIx = 4x, FIx = [1, 2x]$ and for $R > 3$ we can see that $M(IIx, FIx, t) \geq M(Ix, Fx, t/R)$ for all $x \in X$. Thus I and F are weakly commuting of type (Kh) on X but there exists no sequence x_n in X such that condition of δ- compatibility is satisfied.

Example 2.3 Let $X = [1, \infty)$. Define $I : X \to X$ and $F : X \to B(X)$ by $Ix = 2x$ and $Fx = [1, x+1]$ for all $x \in X$. Then $IIx = 4x, FIx = [1, 2x+1]$ and for $R > 3$ we can see that $\delta(IIx, FIx, C) < R\delta(Ix, Fx, C)$ for all $x \in X$. Thus I and F are weakly commuting of type (Kh) on X. On the other hand if we take $x = 1$, thus $I(1) = 2 \in F(1) = [1, 2]$, $IF(1) \neq FI(1)$. Then I and F are not weakly compatible.

3 Main Results

Now we can introduce our main theorems, let $CB(X)$ be the class of all nonempty bounded closed subset of X and $\delta(A, B) = sup\{d(a, b) : a \in A, b \in B\}$.

Theorem 3.1 Let S, R, H and T be four self mappings of a fuzzy metric space $(X, M, *)$ and $A, B : X \to CB(X)$ set-valued mappings satisfying following conditions:

(1) $\bigcup A(X) \subseteq SR(X)$ and $\bigcup B(X) \subseteq TH(X)$,

(2) $\{A, TH\}$ and $\{B, SR\}$ are weakly commuting of type (Kh) at coincidence points in X,

(3) $aM(THx, SRy, t) + bM(THx, Ax, t) + cM(SRy, By, t)$
$+ max\{M(Ax, SRy, t), M(By, THx, t)\} \leq qM(Ax, By, t)$,

for all $x, y \in X$, where $a, b, c \geq 0$ with $0 < q < a + b + c < 1$ and if the range of one of the mappings A, B, SR and TH is complete subspace of X. Then A, B, S, R, H and T have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. From the condition (1), we chose a point x_1 in X such that $SRx_1 \in Ax_0$. For this point x_1 there exist a point x_2 in X such that $THx_2 \in Bx_1$ and so on. Inductively, we can define a sequence $\{Z_n\}$ in X such that
$SRx_{2n+1} \in Ax_{2n} = Z_{2n}, THx_{2n+2} \in Bx_{2n+1} = Z_{2n+1}, \forall\, n = 0, 1, 2, ...$
We will prove that $\{Z_n\}$ is Cauchy sequence.
Now, we prove that $M(Z_{2n+1}, Z_{2n}, t) > M(Z_{2n}, Z_{2n-1}, t)$. Using inequality (3), we obtain
$qM(Z_{2n}, Z_{2n+1}, t) = qM(Ax_{2n}, Bx_{2n+1}, t)$
$\geq aM(THx_{2n}, SRx_{2n+1}, t) + bM(THx_{2n}, Ax_{2n}, t) + cM(SRx_{2n+1}, Bx_{2n+1}, t)$
$+ max\{M(Ax_{2n}, SRx_{2n+1}, t), M(Bx_{2n+1}, THx_{2n}, t)\}$

$\geq aM(Z_{2n-1}, Z_{2n}, t) + bM(Z_{2n-1}, Z_{2n}, t) + cM(Z_{2n}, Z_{2n+1}, t)$
$+ max\{M(Z_{2n}, Z_{2n}, t), M(Z_{2n+1}, Z_{2n-1}, t)\}$.
Then $M(Z_{2n}, Z_{2n+1}, t) \geq \beta M(Z_{2n-1}, Z_{2n}, t)$, where $\beta = \frac{a+b+1}{q-c} > 1$
Since $\beta > 1$, we obtain

$$M(Z_{2n+1}, Z_{2n}, t) > M(Z_{2n}, Z_{2n-1}, t)$$

Similarly

$$M(Z_{2n+2}, Z_{2n+1}, t) > M(Z_{2n+1}, Z_{2n}, t).$$

Now for any positive integer p,
$M(Z_n, Z_{n+p}, t) \geq M(Z_n, Z_{n+1}, \frac{t}{p}) * M(Z_{n+1}, Z_{n+2}, \frac{t}{p}) * ... * M(Z_{n+p-1}, Z_{n+p}, \frac{t}{p})$.
As $n \to \infty$, we get $M(Z_n, Z_{n+p}, t) \to 1$.
Hence $\{Z_n\}$ is a Cauchy sequence. Suppose that SRX is complete, therefore by the above, $\{SRx_{2n+1}\}$ is a Cauchy sequence and hence $SRx_{2n+1} \to z = SRv$ for some $v \in X$. Hence, $Z_n \to z$ and the subsequences THx_{2n+2}, Ax_{2n} and Bx_{2n+1} converge to z.
We shall prove that $z = SRv \in Bv$, by (3), we have
$qM(Ax_{2n}, Bv, t) \geq aM(THx_{2n}, SRv, t) + bM(THx_{2n}, Ax_{2n}, t) + cM(SRv, Bv, t)$

$$+max\{M(Ax_{2n}, SRv, t), M(Bv, THx_{2n}, t)\}.$$

As $n \to \infty$, we obtain
$qM(z, Bv, t) \geq aM(z, z, t) + bM(z, z, t) + cM(z, Bv, t) + max\{M(z, z, t), M(Bv, z, t)\}$
$M(z, Bv, t) \geq (\frac{a+b+1}{q-c}) > 1$,
which yields $\{z\} = \{SRv\} = Bv$.
Since $\bigcup B(X) \subseteq TH(X)$, thus, there exist $u \in X$ such that $\{THu\} = Bv = \{z\} = \{SRv\}$.
Now if $Au \neq Bv$,we get
$qM(Au, Bv, t) \geq aM(THu, SRv, t) + bM(THu, Au, t) + cM(SRv, Bv, t)$

$$+max\{M(Au, SRv, t), M(Bv, THu, t)\},$$

$qM(Au, z, t) \geq aM(z, z, t) + bM(z, Au, t) + cM(z, z, t) + max\{M(Au, z, t), M(z, z, t)\}$,
$M(Au, z, t) \geq (\frac{a+c+1}{q-b}) > 1$,
which yields $Au = \{z\} = \{THu\} = \{SRv\} = Bv$.
Since $Au = \{THu\}$ and $\{A, TH\}$ is weakly commuting of type (Kh) at coincidence points in X, $M(THTHu, ATHu) \geq RM(THu, Au)$ which gives $Az = \{Tz\}$.
On using (3), we obtain
$qM(Az, Bv, t) \geq aM(THz, SRv, t) + bM(THz, Az, t) + cM(SRv, Bv, t)$

$$+max\{M(Az, SRv, t), M(Bv, THz, t)\},$$

$qM(Az, z, t) \geq aM(Tz, z, t) + bM(z, Az, t) + cM(z, z, t) + max\{M(Az, z, t), M(z, z, t)\}$.
Hence, $Az = \{z\} = \{THz\}$. Similarly, $Bz = \{z\} = \{SRz\}$ where $\{B, SR\}$ is

weakly commuting of type (Kh) at coincidence points in X. Then,
$Az = \{THz\} = \{z\} = \{SRz\} = Bz$. Now, we prove that $Rz = z$. In fact, by (3), it follows that

$$qM(Az, BRz, t) \geq aM(THz, SRRz, t) + bM(THz, Az, t) + cM(SRRz, BRz, t) + max\{M(Az, SRRz, t), M(BRz, THz, t)\}.$$

Since $Bz = \{z\} = \{SRz\}$ and $R : X \to X$, thus $BRz = \{Rz\}, SRRz = Rz$.
Then, the above inequality become

$$qM(z, Rz, t) \geq aM(z, Rz, t) + bM(z, z, t) + cM(Rz, Rz, t) + max\{M(z, Rz, t), M(Rz, z, t)\}.$$

Thus, we have $Rz = z$. Hence $Rz = SRz = Sz = z$. Similarly, we get $Tz = Hz = z$. Thus
$Az = \{Tz\} = \{Hz\} = \{z\} = \{Sz\} = \{Rz\} = Bz$.
i.e.,z is the common fixed point of A, B, S, R, H and T have a unique.
To see z is unique, suppose that $p \neq z$ such that $Ap = \{Tp\} = \{p\} = \{Sp\} = Bp$.
On using (3), we get

$$qM(Az, Bp, t) \geq aM(THz, SRp, t) + bM(THz, Az, t) + cM(SRp, Bp, t) + max\{M(Az, SRp, t), M(Bp, THz, t)\},$$

$M(z, p, t) \geq (\frac{b+c}{q-a-1})$,
which is impossible, $z = p$.Then A, B, S, R, H and T have a unique common fixed point.

Remark 3.1 Theorem 3.1 is generalized,extended and improved for results of Abd-Rabou [9] in fuzzy metric space.

Theorem 3.2 Let S and T be two self mappings of a fuzzy metric space $(X, M, *)$ such that

$$(1) aM(Tx, Sy, t) + bM(Tx, x, t) + cM(Sy, y, t) + max\{M(x, Sy, t), M(y, Tx, t)\} \leq qM(x, y, t),$$

for all $x, y \in X$, where $a, b, c \geq 0$ with $0 < q < a + b + c < 1$ and if the range of one of the mappings S and T is complete subspace of X. Then S and T have a unique common fixed point.

Proof. If we set $A = B = H = R = I$(:the identity mapping)in Theorem 3.1, then it is easy to check that the pairs (I, S) and (I, T) are weakly commuting of type (Kh). Hence, by Theorem 3.1, S and T have a unique common fixed point.

In the following theorem, we prove a common fixed point theorem for four self mappings without the continuity assumption of the mappings in Pathak and Singh [5] and Som [13]. Also, we replacing complete fuzzy metric space

$(X, M, *)$ by the range of one of the mappings is complete subspace of X.
Theorem 3.3 Let A, B, S and T are four self mappings of a fuzzy metric space $(X, M, *)$ such that

(1) $A(X) \subseteq S(X)$ and $B(X) \subseteq T(X)$,

(2) $\{A, T\}$ and $\{B, S\}$ are weakly commuting of type (Kh),

(3) $aM(Tx, Sy, t) + bM(Tx, Ax, t) + cM(Sy, By, t)$
$+ max\{M(Ax, Sy, t), M(By, Tx, t)\} \leq qM(Ax, By, t)$,

for all $x, y \in X$, where $a, b, c \geq 0$ with $0 < q < a+b+c < 1$ and if the range of one of the mappings A, B, S and T is complete subspace of X. Then A, B, S and T have a unique common fixed point.
Proof. If we set $A, B : X \to X$ in Theorem 3.1. Hence proof.

Remark 3.2 Theorem 3.3 is generalized,extended and improved for results of Pathak and Singh [5] in fuzzy metric space.

Remark 3.3 Theorem 3.3 is generalized,extended and improved for results of Sharma and Tiwari [13] in fuzzy metric space.

Theorem 3.4 Let S be a self mapping of a fuzzy metric space $(X, M, *)$ and $A : X \to CB(X)$ set-valued mappings satisfying following conditions:

(1) $\bigcup A^n(X) \subseteq S^m(X)$,

(2) the pairs $\{A^n, S^m\}$ are weakly commuting of type (Kh),

(3) $aM(S^m x, S^m y, t) + bM(S^m x, A^n x, t) + cM(S^m y, A^n y, t)$
$+ max\{M(A^n x, S^m y, t), M(A^n y, S^m x, t)\} \leq qM(A^n x, A^n y, t)$,

for all $x, y \in X$, where $a, b, c \geq 0$ with $0 < q < a + b + c < 1$ and if the range of one of the mappings A^n and S^m is complete subspace of X. Then A and S have a unique common fixed point.
Proof. If we set $A = B = A^n$ and $SR = TH = S^m$ in Theorem 3.1, we get A^n and S^m have a unique common fixed point in X. That is, there exists $z \in X$ such that $A^n(z) = \{S^m(z)\} = \{z\}$. since $A^n(Az) = A(A^n z) = Az$, it follows that Az is a fixed point of A^n and S^m and hence $Az = z$. Similarly, we have $Sz = z$.

Theorem 3.5 Let S and T be two self mappings of a fuzzy metric space $(X, M, *)$ and two sequences set-valued mappings $A_i, B_j : X \to CB(X)$ for all $i, j \in N$ satisfying following conditions:

(1) there exists$i_0, j_0 \in N$ such that $\bigcup A_{i_0}(X) \subseteq S(X)$ and $\bigcup B_{j_0}(X) \subseteq T(X)$,

(2) $\{A_{i_0}, T\}$ and $\{B_{j_0}, S\}$ are weakly commuting of type (Kh) pairs,

(3) $aM(Tx, Sy, t) + bM(Tx, A_i x, t) + cM(Sy, B_j y, t)$
$+ max\{M(A_i x, Sy, t), M(B_j y, Tx, t)\} \leq qM(A_i x, B_j y, t)$,

for all $x, y \in X$, where $a, b, c \geq 0$ with $0 < q < a + b + c < 1$ and if the range of one of the mappings A_i, B_j, S and T for all $i, j = 1, 2, ...$ is complete subspace of X. Then A_i, B_j, S and T have a unique common fixed point for all $i, j = 1, 2, ...$

Proof. By Theorem 3.1, the mappings A_{i_0}, B_{j_0}, S and T for some $i_0, j_0 \in N$ have a unique common fixed point in X. That is, there exists a unique point $z \in X$ such that
$\{Sz\} = \{Tz\} = \{z\} = A_{i_0}z = B_{j_0}z$.
Suppose that there exists $i \in N$ such that $i \neq i_0$. Then, we have $qM(A_i z, z, t) =$
$qM(A_i x, B_{j_0} z, t)$
$\geq aM(Tz, Sz, t) + bM(Tz, A_i z, t) + cM(Sz, B_{j_0} z, t)$
$+ max\{M(A_i x, Sz, t), M(B_{j_0} z, Tz, t)\}$
$\geq aM(z, z, t) + bM(z, A_i z, t) + cM(z, z, t)$
$+ max\{M(A_i x, z, t), M(z, z, t)\}$
$> (a + b + c + 1)M(z, A_i z, t)$,
which is a contradiction. Hence, for all $i \in N$, it follows that $A_i z = z$. Similarly, for all $j \in N$, we have $B_j z = z$. Therefor, for all $i, j \in N$, we have
$A_i z = B_j z = z = \{Sz\} = \{Tz\}$.

4 Open Problem

We can study common fixed point theorems for six hybrid mappings in fuzzy 2-metric spaces, without assuming compatibility and continuity of any mapping on non complete fuzzy 2-metric spaces. we can use a non compatible condition, that is, weak commutativity of type (Kh) in fuzzy 2-metric spaces. We can show that completeness of the whole space is not necessary for the existence and uniqueness of common fixed point. Also, we can prove a common fixed point theorem for sequences of mappings by the same weaker conditions.

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