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Explicit Solutions of Fractional Schrödinger Equation via Fractional Calculus Operators

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Abstract

In this paper, we investigate the schrödinger equation in a given α - dimensional fractional space with a columb potential depending on a parameter and obtain explicit solution of second order linear ordinary differential equation.

Keywords: *Fractional calculus; Schrödinger equation; Radial equation; Generalized Leibniz rule; Ordinary differential equation*

AMS subject classifications: 26A33, 34A08

1 Introduction

Fractional differential equations have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Fractional calculus is "the theory of derivatives and integrals of any arbitrary real or complex order, which unify and generalize the notions of integer-order differentiation and n-fold integration" [1-3]. It has been in the minds of mathematicians for 315 years and still contains many questions. Firstly, the idea of this area appeared in a letter by Leibniz to L' Hospital in (1695). In the following three hundred years a lot of mathematicians contribute to the fractional calculus: Johann Bernoulli , John Wallis, L. Euler, J.L. Lagrange, P.S. Laplace, S.F. Lacroix, J.B.J. Fourier, N.H. Abel, J. Liouville, S.S. Greatheed, A.De Morgan, B. Riemann, W. Center, H. Holmgren, A.K. Grünwald, A.V. Letnikov, H. Laurent, O. Heaviside, G.H. Hardy, H. Weyl, E.L. Post, H.T. Davis, A. Erdélyi, H. Kober, A. Zygmund, M. Riesz, I.M. Gel'fand, G.E. Shilov, I.N.

Sneddon, S.G. Samko, T.J. Osler, E.R. Love, and many others [4,5]. In last decades, fractional calculus has been the concept of ever increasing interest because of its applications in physics and engineering. The differrintegration operators and their generalizations [6,7,8] have been used to solve some classes of differential equations and fractional differential equations.

Definition 1.1 *Let* $D=\{D^-,D^+\}, C=\{C^-,C^+\}$ *where* C^- *is a curve along the cut joing two points* z *and* $-\infty+iIm(z)$, C^+ *is a curve along the cut joing two points* z *and* $\infty+iIm(z)$, D^- *is a domain surrounded by* C^-, *and* D^+ *is a domain surrounded by* C^+ *(Here* D *contains the points over the curve* C *). Moreover, let* $f=f(z)$ *be a regular function in* D $(z\in D)$ *such that*

$$f_\mu(z)=(f(z))_\mu=\frac{\Gamma(\mu+1)}{2\pi i}\int_C\frac{f(t)}{(t-z)^{\mu+1}}dt,\quad (\mu\neq-1,-2,...) \tag{1}$$

and

$$f_{-n}(z)=\lim_{\mu\to-n}f_\mu(z)\quad (n\in \mathrm{Z}^+), \tag{2}$$

where $t\neq z$,

$$-\pi\leq\arg(t-z)\leq\pi\quad \textit{for } C^-$$

and

$$0\leq\arg(t-z)\leq 2\pi\quad \textit{for } C^+.$$

Then $f_\mu(z)(\mu>0)$ *is said to be the fractional derivative of* $f(z)$ *of order* μ *and* $f_\mu(z)(\mu<0)$ *is said to be the fractional integral of* $f(z)$ *of order* $-\mu$, *provided (in each case) that*

$$|f_\mu(z)|<\infty\quad(\mu\in R). \tag{3}$$

It is worth to recall the following useful lemmas and properties associated with the fractional differintagration defined above [6,9].

Lemma 1.1 *(Linearity). Let* $f(z)$ *and* $g(z)$ *be analytic and single-valued functions. If* f_μ *and* g_μ *exist, then*

$i)$ $(h_1 f(z))_\mu=h_1 f_\mu(z)$

$ii)$ $(h_1 f(z)+h_2 g(z))_\mu=h_1 f_\mu(z)+h_2 g_\mu(z)$ (4)

hold, where h_1 *and* h_2 *are constants and* $\mu\in R; z\in C$.

Lemma 1.2 *(Index law). Let $f(z)$ be an analytic and single-valued function. If $(f_\rho)_\mu$ and $(f_\mu)_\rho$ exist, then*

$$\left(f_\rho(z)\right)_\mu = f_{\rho+\mu}(z) = \left(f_\mu(z)\right)_\rho, \tag{5}$$

where $\rho, \mu \in R; z \in C$ *and* $\left|\dfrac{\Gamma(\rho+\mu+1)}{\Gamma(\rho+1)\Gamma(\mu+1)}\right| < \infty.$

Lemma 1.3 *(Generalized Leibniz rule). Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If $f_\mu(z)$ and $g_\mu(z)$ exist, then*

$$\left(f(z)g(z)\right)_\mu = \sum_{n=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)\Gamma(n+1)} f_{\mu-n}(z) g_n(z), \tag{6}$$

where $\mu \in R; z \in C$ *and* $\left|\dfrac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)\Gamma(n+1)}\right| < \infty$.

Property 1.4. *For a constant* λ,

$$\left(e^{\lambda z}\right)_\mu = \lambda^\mu e^{\lambda z} \quad (\lambda \neq 0; \mu \in R; z \in C). \tag{7}$$

Property 1.5. *For a constant* λ,

$$\left(e^{-\lambda z}\right)_\mu = e^{-i\pi\mu} \lambda^\mu e^{-\lambda z} \quad (\lambda \neq 0; \mu \in R; z \in C). \tag{8}$$

Property 1.6. *For a constant* λ,

$$\left(z^\lambda\right)_\mu = e^{-i\pi\mu} \frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\mu} \quad \left(\mu \in R; z \in C; \left|\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{9}$$

Some of the most recent studies on the subject of particular solutions of linear ordinary and partial fractional differintegral equations are those given by Tu et al. [10] who presented unification and generalization of a significantly large number of widely scattered results on this subject, involving a family of linear ordinary fractional differintegral equations as follows.

Theorem 1.7. *Let $P(z;p)$ and $Q(z;q)$ be polynomials in z of degrees p and q, respectively, defined by*

$$P(z;p) = \sum_{k=0}^{p} a_k z^{p-k} = a_0 \prod_{j=1}^{p} (z - z_j) \quad (a_0 \neq 0, p \in N) \tag{10}$$

and

$$Q(z;q) = \sum_{k=0}^{q} b_k z^{q-k} \quad (b_0 \neq 0, q \in N). \tag{11}$$

Suppose also that $f_{-\nu}(z) \neq 0$ *exists for a given function* f.

Then the nonhomogeneous linear ordinary fractional differintegral equation

$$P(z;p)\phi_{\mu}(z)+\left[\sum_{k=1}^{p}\binom{\nu}{k}P_k(z;p)+\sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q)\right]\phi_{\mu-k}(z) \\ +\binom{\nu}{q}q!b_0\phi_{\mu-q-1}(z)=f(z) \quad (\mu,\nu\in R,\ p,q\in N) \tag{12}$$

has a particular solution of the form

$$\phi(z)=\left(\left(\frac{f_{-\nu}(z)}{P(z;p)}e^{H(z;p,q)}\right)_{-1}e^{-H(z;p,q)}\right)_{\nu-\mu+1} \quad (z\in C-\{z_1,...,z_p\}, \tag{13}$$

where for convenience,

$$H(z;p,q)=\int^{z}\frac{Q(\xi,q)}{P(\xi,q)}d\xi, \quad (z\in C-\{z_1,...,z_p\}, \tag{14}$$

provided that the second member of (13) exists. Furthermore, the homogeneous linear ordinary fractional differintegral equation

$$P(z;p)\phi_{\mu}(z)+\left[\sum_{k=1}^{p}\binom{\nu}{k}P_k(z;p)+\sum_{k=1}^{q}\binom{\nu}{k-1}Q_{k-1}(z;q)\right]\phi_{\mu-k}(z) \\ +\binom{\nu}{q}q!b_0\phi_{\mu-q-1}(z)=0 \quad (\mu,\nu\in R,\ p,q\in N) \tag{15}$$

has solutions of the form

$$\phi(z)=K\left(e^{-H(z;p,q)}\right)_{\nu-\mu+1} \tag{16}$$

where K *is an arbitrary constant and* $H(z;p,q)$ *is given by (14), it being provided that the second member of (16) exist* [10].

2 Main Results

We consider the Schrödinger equation

$$\left[-\frac{\hbar^2}{2mr^{\alpha-1}}\frac{\partial}{\partial r}\left(r^{\alpha-1}\frac{\partial}{\partial r}\right)+\frac{\ell^2}{2mr^2}-e^2\frac{\kappa}{r^{\delta-2}}\right]\varphi(r,\theta)=\left(E-E_g\right)\varphi(r,\theta), \tag{17}$$

where ℓ^2 corresponds to the angular momentum operator given by

$$\ell^2\varphi(r,\theta)=\left[-\frac{\hbar^2}{\sin^{\alpha-2}}\frac{\partial}{\partial\theta}\left(\sin^{\alpha-1}\frac{\partial}{\partial\theta}\right)\right]\varphi(r,\theta)=\ell(\ell+\alpha-2)\varphi(r,\theta), \quad (18)$$

Where α is the dimension of a solid $(1\le\alpha\le3)$, and the radial interval r $(0\le r\le\infty)$ and related angle $\theta(0\le\theta\le\pi)$ measured relative to an axis passing through the origin are two coordinates describing r in the α-dimensional space. The constant κ has the value of $\frac{1}{4\pi\varepsilon_0}$ for $\delta=3$ and is generally showed as [11]

$$\kappa=\frac{\Gamma\left(\frac{\delta}{2}\right)}{2\pi^{\delta/2}(\delta-2)\varepsilon_0}\quad(\delta>2). \quad (19)$$

By means of equation (17) in the form

$$\varphi(r,\theta)=R(r)\Phi(\theta).$$

We can obtain that

$$R''(r)+\frac{\alpha-1}{r}R'(r)+\left[\frac{2m}{\hbar^2}\left((E-E_g)+e^2\frac{\kappa}{r^{\delta-2}}\right)-\frac{\ell(\ell-\alpha-2)}{r^2}\right]R(r)=0, \quad (20)$$

$$\Phi''(\theta)+(\alpha-2)\cot\theta\Phi'(\theta)+\ell(\ell-\alpha-2)\Phi(\theta)=0. \quad (21)$$

The following equality is solutions by aid of Gegenbauer polynomials $C_\ell^{(\alpha/2)-1}(\cos\theta)$ for the angular equation (18)

$$\Phi_\ell(\theta)=H_\ell(\alpha)C_\ell^{(\alpha/2)-1}(\cos\theta)\quad(\ell=0,1,2,...,n-1), \quad (22)$$

where H_ℓ is the normalization factor and given by [12],

$$H_\ell(\alpha)=\begin{cases}\Gamma\left(\frac{\alpha}{2}-1\right)\left[\frac{\ell!\left(\ell+\frac{\alpha}{2}-1\right)}{2^{3-\alpha}\pi\Gamma(\ell+\alpha-2)}\right]^{1/2} & (\alpha\ne2),\\ \frac{1}{(2\pi)^{1/2}}\quad(\ell\ne0)\ \ or\ \ \frac{1}{2\pi^{1/2}} & (\ell=0)(\alpha=2).\end{cases} \quad (23)$$

Solving the radial equation $R(r)$, we need to use the substitutions

$$R(r) = r^{\ell} e^{-kr} \phi(r), \tag{24}$$

where $k^2 = -\frac{2m(E - E_g)}{\hbar^2}$.

We find the following differential equation

$$z\phi''(z) + \left[(2\ell + \alpha - 1) - z\right]\phi'(z) + \left[\frac{b}{2^{3-\delta} k^{4-\delta}} z^{3-\delta} - \frac{2\ell + \alpha - 1}{2}\right]\phi(z) = 0, \tag{25}$$

by using the substitutions

$$z = 2kr, \quad b = \frac{me^2\kappa}{\hbar^2}. \tag{26}$$

We obtain at the special case as given in reference for $\delta = 3$ [12].

Let us consider the differential equation

$$z\frac{d^2\phi}{dz^2} + (\tau - z)\frac{d\phi}{dz} + \left(\sigma z^{3-\delta} - \frac{\tau}{2}\right)\phi(z) = 0, \tag{27}$$

where

$$\tau = 2\ell + \alpha - 1, \quad \sigma = \frac{b}{2^{3-\delta} k^{4-\delta}}. \tag{28}$$

Let $\delta = 3$. For this δ the equation (27) becomes the differential equation

$$z\frac{d^2\phi}{dz^2} + (\tau - z)\frac{d\phi}{dz} + \left(\sigma - \frac{\tau}{2}\right)\phi(z) = 0. \tag{29}$$

For equation (29), using the substitution

$$\phi(z) = z^{-\tau/2} e^{z/2} u(z). \tag{30}$$

Thus, we have

$$\phi'(z) = z^{-\frac{\tau}{2}-1} e^{z/2}\left[z\frac{du}{dz} + \frac{1}{2}(z - \tau)u(z)\right], \tag{31}$$

and

$$\phi''(z) = z^{-\frac{\tau}{2}-2} e^{z/2}\left\{z^2\frac{d^2u}{dz^2} + z(z - \tau)\frac{du}{dz} + \frac{1}{4}\left[(z - \tau)^2 + 2\tau\right]u(z)\right\}. \tag{32}$$

After substituting $\phi(z), \phi'(z)$ and $\phi''(z)$ into (29), performing necessary operations, we obtain at the differential equation

$$\frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{\sigma}{z} + \frac{2\tau - \tau^2}{4z^2}\right)u(z) = 0. \tag{33}$$

We can write the last equation in the form

$$\frac{d^2u}{dz^2}+\left[-\frac{1}{4}+\frac{\sigma}{z}+\frac{\frac{1}{4}-\left(\frac{\tau-1}{2}\right)^2}{z^2}\right]u(z)=0. \tag{34}$$

For the problem having the analogous singularity, some questions of spectral analysis are given in [13].

Using Theorem 1.7,

$$\mu=2,\quad p=q=1,\quad a_0=h\neq 0,\quad a_1=0,\quad b_0=n\neq 0,\quad b_1=s, \tag{35}$$

so that

$$P(z;1)=hz,\quad P_1(z;1)=h, \tag{36}$$

and

$$Q(z;1)=nz+s,\quad Q_1(z;1)=n. \tag{37}$$

Therefore, we obtain from definition (14) that

$$H(z;1,1)=\int^z \frac{Q(\xi;1)}{P(\xi;1)}d\xi$$

$$=\int^z \frac{n\xi+s}{h\xi}d\xi$$
$$=\ln\left[(hz)^{s/h}e^{nz/h}\right]. \tag{38}$$

By substituting from (35) to (38) into Theorem 1.7, we can find the following relevant application of Theorem 1.7.

Theorem 2.1 *The homogeneous second order linear ordinary differential equation*

$$hz\frac{d^2\phi}{dz^2}+(nz+\nu h+s)\frac{d\phi}{dz}+\nu n\phi(z)=0\quad (h\neq 0,\nu\in R), \tag{39}$$

has a solution of the form

$$\phi(z) = K\left[(hz)^{-s/h} e^{-nz/h}\right]_{\nu-1}, \tag{40}$$

where K is an arbitrary constant, provide that the right hand of (40) exists.

Now, in Theorem 2.1, we further set

$$h = 1, \quad n = -1, \quad s = \frac{\tau}{2} + \sigma, \quad \nu = \frac{\tau}{2} - \sigma. \tag{41}$$

We thus obtain that the homogeneous linear ordinary differential equation

$$z\frac{d^2\phi}{dz^2} + (\tau - z)\frac{d\phi}{dz} + \left(\sigma - \frac{\tau}{2}\right)\phi(z) = 0 \quad \left(z \in C - \{0\}\right),$$

has a solution of the form

$$\phi(z) = K\left[z^{-\left(\frac{\tau}{2}+\sigma\right)} e^{z}\right]_{\frac{\tau}{2}-\sigma-1}. \tag{42}$$

Thus, the homogeneous linear ordinary differential equation (34) has a solution given by

$$\begin{aligned} u(z) &= z^{\tau/2} e^{-z/2} \phi(z) \\ &= K z^{\tau/2} e^{-z/2} \left(z^{-\left(\frac{\tau}{2}+\sigma\right)} e^{z}\right)_{\frac{\tau}{2}-\sigma-1}. \end{aligned} \tag{43}$$

Example 2.2 If we substitute $\tau = 2, \sigma = -1$ in equation (34), then we obtain the following equation,

$$u_2 - \left(\frac{1}{4} + \frac{1}{z}\right)u = 0, \tag{44}$$

the solution is

$$u = Kze^{-\frac{z}{2}}\left(e^{z}\right)_1. \tag{45}$$

By performing necassary operations in (45), we get

$$u_2 = Kze^{z/2}\left(\frac{z}{4} + 1\right). \tag{46}$$

If equality (45), (46) put into (44), we can see easily that is a solution of (44).

3 Conclusion

Several authors demonstrated the usefulness of fractional calculus operators in the derivation of particular solutions of a considerably large number of linear ordinary and partial differential equations of the second and higher orders. By means of fractional calculus techniques, we find explicit solutions of second order linear ordinary differential equations.

4 Open Problem

In this work, we obtain explicit solution for the schrödinger equation with a columb potential. The method can be applied different for potentials.

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