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On Sequence Spaces of Invariant Means and Lacunary Defined by a Sequence of Orlicz Functions

Çiğdem A. Bektaş[1] and Gülcan Atıci[2]

Department of Mathematics, Firat University, Elazig, 23119, TURKEY.
e-mail:cigdemas78@hotmail.com
Department of Mathematics, Muş Alparslan University, Muş, 49100, TURKEY.
e-mail: gatici23@hotmail.com

Abstract

The purpose of this paper is to introduce and study some sequence spaces which are defined by combining the concepts of a sequence of Orlicz functions, invariant mean and lacunary convergence. We establish some inclusion relations between these spaces under some conditions.

Keywords: *Difference sequence, sequence of Moduli, lacunary sequence.*

1 Introduction

One hundred years ago mathematics was undergoing a revolution. The Kantian dictate that Euclidean Geometry is the only rationally conceivable basis for the physical universe had been debunked. Numerous alternative geometries, each self-consistent, were being discovered, axiomatized, and developed. Felix Klein found a unifying principle for relating and classifying the various geometries Invariant Theory. The key idea is to classify mathematical structures by the transformations under which they are invariant. Invariant Theory has achieved wide influence in mathematics, physics (including relativity and quantum mechanics), and computer science. The calculus developed here is based upon relatively simple aspects of Invariant Theory.

From a mathematical point of view, transition from classical mechanics to quantum mechanics amounts to, among other things, passing from the commutative algebra of classical observables to the non-commutative algebra of quantum mechanical observables. Recall that in classical mechanics an observable (e.g. energy, position, momentum, etc.) is a function on a manifold called the phase space of the system. A little more than 50 years after these developments, Alain Connes realized that a similar procedure can in fact be applied to areas of mathematics where the classical notions of space (e.g. measure space, locally compact space, or a smooth space) loses its applicability and pertinence and can be replaced by a new idea of space, represented by a non-commutative algebra. Conne's theory, which is generally known as non-commutative geometry, is a rapidly growing new area of mathematics that interacts with and contributes to many disciplines in mathematics and physics. For a recent survey, see Conne's article [3].

Examples of such interactions and contributions include: theory of operator algebras, index theory of elliptic operators, algebraic and differential topology, number theory, standard model of elementary particles, quantum Hall effect, renormalization in quantum field theory and string theory.

As cited above operator algebras are presently one of the dynamic areas of mathematics.

Invariant means on amenable groups are an important tool in many parts of mathematics, especially in harmonic analysis invariant means and their generalizations for vector-valued functions play also an important role in the stability of functional equations and selections of set-valued functions (see, for example, [8,9]). Thus it seems natural to ask what are possible limitations of the use of invariant means. We will show that invariant means are, in some sense, naturally restricted to reflexive Banach spaces. (see [9])

In this paper, by introducing some sequence spaces which are related to a sequence of Orlicz functions, invariant mean, lacunary convergence, we establish some inclusion relations between these spaces under some conditions.

Nowadays operator algebra, operator theory and lacunary convergence play an important role in different areas of mathematics, and its applications, particularly in Mathematics, Physics and Numerical analysis.

It is hoped that this study about operator theory serves for researchers who carry research in various fields of science.

Let ℓ_∞ and c denote the Banach spaces of bounded and convergent sequences $x = (x_k)$, with $x_k \in \mathbb{R}$ or $\mathbb{C}$, normed by $\|x\| = \sup_k |x_k|$, respectively.

The difference sequence spaces was first introduced by Kızmaz [11] and then the concept was generalized by Et and Çolak [5]. Later on, Et and Esi [6] extended the difference sequence spaces to the sequence spaces

$$X\left(\Delta_v^m\right) = \left\{x = (x_k) : \left(\Delta_v^m x\right) \in X\right\},$$

for $X = \ell_\infty$, c or c_0, where $v = (v_k)$ be any fixed sequence of non-zero complex numbers and $(\Delta_v^m x_k) = (\Delta_v^{m-1} x_k - \Delta_v^{m-1} x_{k+1})$, $\Delta_v^m x_k =_{i=0}^{m} (-1)^i \binom{m}{i} v_{k+i} x_{k+i}$ for all $k \in \mathbb{N}$.

The sequence spaces $\Delta_v^m(\ell_\infty)$, $\Delta_v^m(c)$ and $\Delta_v^m(c_0)$ are Banach spaces normed by

$$\|x\|_\Delta =_{i=1}^{m} |v_i x_i| + \|\Delta_v^m x\|_\infty .$$

Let σ be a mapping of the set of positive integers into itself. A continuous linear functional ϕ on ℓ_∞, is said to be an invariant mean or $\sigma-$mean if and only if

(i) $\phi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\phi(e) = 1$, $e = (1, 1, ...)$

(iii) $\phi(x_{\sigma(n)}) = \phi(x)$ for all $x \in \ell_\infty$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{ x \in \ell_\infty : \lim_k t_{kn}(x) = l, \text{ uniformly in } n \right\},$$

$l = \sigma - \lim x$ where

$$t_{kn}(x) = \frac{x_n + x_{\sigma^1(n)} + x_{\sigma^2(n)} + ... + x_{\sigma^k(n)}}{k+1} \text{ [14].}$$

In the case σ is the translation mapping $n \to n+1$, $\sigma-$mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequence (see [12]).

By a lacunary sequence $\theta = (k_r); r = 0, 1, 2, ...$ where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [7] as

$$N_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r}{}_{k \in I_r} |x_k - l| = 0, \text{ for some } l \right\}.$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 \leq \lambda \leq 1$.

Lindenstrauss and Tzafriri [13] used the idea of Orlicz function to define what is called an Orlicz sequence space

$$\ell_M = \left\{ x \in w :_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}$$

which is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 :_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho} \right) \leq 1 \right\}.$$

Definition 1.1 *Any two Orlicz functions M_1 and M_2 are said to be equivalent if there are positive constant α and β, and x_0 such that $M_1(\alpha x) \leq M_2(x) \leq M_1(\beta x)$ for all x with $0 \leq x \leq x_0$, [10].*

Definition 1.2 *A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$, [10].*

Definition 1.3 *A sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces, [10].*

Remark 1. From the two above definitions it is clear that "A sequence space E is solid implies that E is monotone", [2].

The following inequality will be used throughout the article. Let $p = (p_k)$ be a positive sequence of real numbers with $0 < p_k \leq \sup p_k = G$, $D = \max(1, 2^{G-1})$. Then for all $a_k, b_k \in \mathbb{C}$ for all $k \in N$, we have

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}. \tag{1}$$

2 Main Results

Definition 2.1 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions, $p = (p_k)$ be any sequence of strictly positive real numbers and $u = (u_k)$ be a sequence of positive real numbers . Then we define the following sequence spaces:*

$$[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m) = \left\{x = (x_k) : \sup_{r,n} \frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho}\right)\right]^{p_k} < \infty\right\}$$

$$[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m) = \left\{x = (x_k) : \lim_{r} \frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k - le)|}{\rho}\right)\right]^{p_k} = 0\right\}$$

$$[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m) = \left\{x = (x_k) : \lim_{r} \frac{1}{h_r} \sum_{k \in I_r} u_k \left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho}\right)\right]^{p_k} = 0\right\}$$

where for some $\rho > 0$ and uniformly in n.

Some well-known spaces are obtained by specializing M_k, u, v and m:

If $u = (u_k) = (1, 1, ...)$ for all $k \in \mathbb{N}$, then $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m) = [w^\theta, \mathcal{M}, p]_\sigma^\infty(\Delta_v^m)$, $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m) = [w^\theta, \mathcal{M}, p]_\sigma(\Delta_v^m)$ and $[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m) = [w^\theta, \mathcal{M}, p]_\sigma^0(\Delta_v^m)$.

If $M_k = M$ for all $k \in \mathbb{N}$ and $u = (u_k) = (1, 1, ...)$ for all $k \in \mathbb{N}$, then we have the sequence spaces $[w^\theta, M, p]_\sigma^\infty(\Delta_v^m)$, $[w^\theta, M, p]_\sigma(\Delta_v^m)$ and $[w^\theta, M, p]_\sigma^0(\Delta_v^m)$.

If $M_k(x) = x$ for all $k \in \mathbb{N}$ and $u = (u_k) = (1, 1, ...)$ for all $k \in \mathbb{N}$, then we have the sequence spaces $[w^\theta, p]_\sigma^\infty(\Delta_v^m)$, $[w^\theta, p]_\sigma(\Delta_v^m)$ and $[w^\theta, p]_\sigma^0(\Delta_v^m)$.

If $m = 0$ and $v = (v_k) = (1, 1, ...)$ for all $k \in \mathbb{N}$, then we obtain $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty$, $[w^\theta, \mathcal{M}, p, u]_\sigma$ and $[w^\theta, \mathcal{M}, p, u]_\sigma^0$ instead of $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m)$, $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m)$ and $[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m)$, respectively.

Theorem 2.2 *$[w^\theta, \mathcal{M}, p, u]^\infty_\sigma(\Delta^m_v)$, $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta^m_v)$ and $[w^\theta, \mathcal{M}, p, u]^0_\sigma(\Delta^m_v)$ are linear space over the field of complex numbers.*

Proof is trivial.

Theorem 2.3 *$[w^\theta, \mathcal{M}, p, u]^0_\sigma(\Delta^m_v)$ is a topological linear space, total paranormed by*

$$g(x) = \inf\left\{\rho^{p_r/H} : \left(\frac{1}{h_r}\sum_{k\in I_r} u_k\left[M_k\left(\frac{|t_{kn}(\Delta^m_v x_k)|}{\rho}\right)\right]^{p_k}\right)^{1/H} \leq 1, r \geq 1, m = 1, 2, ...\right\}$$

where $H = \max(1, \sup p_k)$.

The proof is routine verification by using standard arguments and therefore omitted.

Theorem 2.4 *Let M be an Orlicz function, then $[w^\theta, M, p]^0_\sigma(\Delta^m_v) \subset [w^\theta, M, p]_\sigma(\Delta^m_v)$ $\subset [w^\theta, M, p]^\infty_\sigma(\Delta^m_v)$.*

Proof. The first inclusion is obvious. We establish the second inclusion. Let $x \in [w^\theta, M, p]_\sigma(\Delta^m_v)$. Then there exists some positive number ρ_1 such that

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(\Delta^m_v x_k - le)|}{\rho_1}\right)\right]^{p_k} \to 0$$

as $r \to \infty$ uniformly in n. Define $\rho = 2\rho_1$. Since M is non-decreasing and convex, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(\Delta^m_v x_k)|}{\rho}\right)\right]^{p_k}$$

$$\leq \frac{D}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(\Delta^m_v x_k - le)|}{\rho_1}\right)\right]^{p_k} + \frac{D}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|le|}{\rho_1}\right)\right]^{p_k}$$

$$\leq \frac{D}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{kn}(\Delta^m_v x_k - le)|}{\rho_1}\right)\right]^{p_k} + D\max\left\{1, \left[M\left(\frac{|le|}{\rho_1}\right)\right]^{G}\right\}$$

by (1). Thus $x \in [w^\theta, M, p]^\infty_\sigma(\Delta^m_v)$.

Theorem 2.5 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. If $\sup_k[M_k(z)]^{p_k}$ $< \infty$ for all $z > 0$, then*

$$[w^\theta, \mathcal{M}, p]_\sigma(\Delta^m_v) \subset [w^\theta, \mathcal{M}, p]^\infty_\sigma(\Delta^m_v).$$

Proof. Let $x \in [w^\theta, \mathcal{M}, p]_\sigma(\Delta_v^m)$. By using (1), we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho}\right)\right]^{p_k} \leq \frac{D}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k - le)|}{\rho}\right)\right]^{p_k} + \frac{D}{h_r}\sum_{k\in I_r}\left[M_k\left(\frac{|le|}{\rho}\right)\right]^{p_k}.$$

Since $\sup_k [M_k(z)]^{p_k} < \infty$, we can take that $\sup_k [M_k(z)]^{p_k} = K$. Hence we get $x \in [w^\theta, \mathcal{M}, p]_\sigma^\infty(\Delta_v^m)$. This completes the proof.

The proofs of the following theorems are obtained by using the same technique of Bektaş [1], therefore we give it without proof.

Theorem 2.6 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Then the following statements are equivalent:*
(i) $[w^\theta, p]_\sigma^\infty(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p]_\sigma^\infty(\Delta_v^m)$,
(ii) $[w^\theta, p]_\sigma^0(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p]_\sigma^\infty(\Delta_v^m)$,
(iii) $\sup_r \frac{1}{h_r}\sum_{k\in I_r}[M_k(z/\rho)]^{p_k} < \infty$ *for all* $z, \rho > 0$.

Theorem 2.7 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. Then the following statements are equivalent:*
(i) $[w^\theta, \mathcal{M}, p]_\sigma^0(\Delta_v^m) \subset [w^\theta, p]_\sigma^0(\Delta_v^m)$,
(ii) $[w^\theta, \mathcal{M}, p]_\sigma^0(\Delta_v^m) \subset [w^\theta, p]_\sigma^\infty(\Delta_v^m)$,
(iii) $\inf_r \frac{1}{h_r}\sum_{k\in I_r}[M_k(z/\rho)]^{p_k} > 0$ *for all* $z, \rho > 0$.

Theorem 2.8 *Let $\mathcal{M} = (M_k)$ be a sequence of Orlicz functions. $[w^\theta, \mathcal{M}, p]_\sigma^\infty(\Delta_v^m) \subset [w^\theta, p]_\sigma^0(\Delta_v^m)$ if and only if*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}[M_k(z/\rho)]^{p_k} = \infty \ (z, \rho > 0).$$

Theorem 2.9 *Let $m \geq 1$ be a fixed integer, then*
(i) $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^{m-1}) \subset [w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m)$,
(ii) $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^{m-1}) \subset [w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m)$,
(iii) $[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^{m-1}) \subset [w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m)$.

Proof. The proof of the inclusions follows from the following inequality

$$\frac{1}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho}\right)\right]^{p_k} \leq \frac{D}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|t_{kn}(\Delta_v^{m-1} x_k)|}{\rho}\right)\right]^{p_k} + \frac{D}{h_r}\sum_{k\in I_r}u_k\left[M_k\left(\frac{|t_{kn}(\Delta_v^{m-1} x_{k+1})|}{\rho}\right)\right]^{p_k}.$$

Theorem 2.10 *Let $\mathcal{M} = (M_k)$ and $\mathcal{T} = (T_k)$ be any two sequence of Orlicz functions. Then we have*
(i) $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m) \cap [w^\theta, \mathcal{T}, p, u]_\sigma^\infty(\Delta_v^m) \subset [w^\theta, \mathcal{M} + \mathcal{T}, p, u]_\sigma^\infty(\Delta_v^m)$,
(ii) $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m) \cap [w^\theta, \mathcal{T}, p, u]_\sigma(\Delta_v^m) \subset [w^\theta, \mathcal{M} + \mathcal{T}, p, u]_\sigma(\Delta_v^m)$,
(iii) $[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m) \cap [w^\theta, \mathcal{T}, p, u]_\sigma^0(\Delta_v^m) \subset [w^\theta, \mathcal{M} + T, p, u]_\sigma^0(\Delta_v^m)$.

Proof. (i) Let $x \in [w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m) \cap [w^\theta, \mathcal{T}, p, u]_\sigma^\infty(\Delta_v^m)$. Then

$$\sup_{r,n} \frac{1}{h_r}{}_{k\in I_r} u_k \left[M_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} < \infty$$

and

$$\sup_{r,n} \frac{1}{h_r}{}_{k\in I_r} u_k \left[T_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} < \infty$$

uniformly in n. We have

$$\left[(M_k + T_k) \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} \leq D \left[M_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} + D \left[T_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k}$$

by (1). Applying ${}_{k\in I_r}$ and multiplying u_k and $\frac{1}{h_r}$ both side of this inequality, we get

$$\begin{aligned} \frac{1}{h_r}{}_{k\in I_r} u_k \left[(M_k + T_k) \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} \leq\ & \frac{D}{h_r}{}_{k\in I_r} u_k \left[M_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} \\ & + \frac{D}{h_r}{}_{k\in I_r} u_k \left[T_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{p_k} \end{aligned}$$

uniformly in n. This completes the proof. (ii) and (iii) can be proved similar to (i).

Theorem 2.11 *Let $\mathcal{M} = (M_k)$ and $\mathcal{T} = (T_k)$ be two sequence of Orlicz functions. If $\mathcal{M}$ and $\mathcal{T}$ are equivalent then*
(i) $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m) = [w^\theta, \mathcal{T}, p, u]_\sigma^\infty(\Delta_v^m)$,
(ii) $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m) = [w^\theta, \mathcal{T}, p, u]_\sigma(\Delta_v^m)$,
(iii) $[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m) = [w^\theta, \mathcal{T}, p, u]_\sigma^0(\Delta_v^m)$.

Proof. Proof follows from Definition 1.1.

Theorem 2.12 *Let $0 < p_k \leq q_k$ for each k and (q_k / p_k) be bounded, then*
(i) $[w^\theta, \mathcal{M}, q]_\sigma^\infty(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p]_\sigma^\infty(\Delta_v^m)$,
(ii) $[w^\theta, \mathcal{M}, q]_\sigma(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p]_\sigma(\Delta_v^m)$,
(iii) $[w^\theta, \mathcal{M}, q]_\sigma^0(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p]_\sigma^0(\Delta_v^m)$.

Proof. Let $x \in [w^{\theta}, \mathcal{M}, p]_{\sigma}^{0}$. Then

$$\sup_{r,n} \frac{1}{h_r}{}_{k\in I_r} \left[M_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{q_k} < \infty$$

uniformly in n. Write $\mu_{k,n} = \left[M_k \left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho} \right) \right]^{q_k}$ and $\lambda_k = p_k / q_k$. Since $p_k \leq q_k$ therefore $0 < \lambda < \lambda_k \leq 1$. Define $y_{k,n} = \mu_{k,n}$, $y_{k,n} = 0$ if $\mu_{k,n} \geq 1$ and $z_{k,n} = \mu_{k,n}$, $z_{k,n} = 0$ if $\mu_{k,n} \geq 1$. So $\mu_{k,n} = y_{k,n} + z_{k,n}$ and $\mu_{k,n}^{\lambda_k} = y_{k,n}^{\lambda_k} + z_{k,n}^{\lambda_k}$.

Now it follows that $y_{k,n}^{\lambda_k} \leq y_{k,n} \leq z_{k,n}$ and $z_{k,n}^{\lambda_k} \leq z_{k,n}^{\lambda}$. Therefore

$$h_r^{-1}{}_{k\in I_r} \mu_{k,n}^{\lambda_k} = h_r^{-1}{}_{k\in I_r} \left(y_{k,n}^{\lambda_k} + z_{k,n}^{\lambda_k} \right) \leq h_r^{-1}{}_{k\in I_r} y_{k,n} + h_r^{-1}{}_{k\in I_r} z_{k,n}^{\lambda}.$$

Since $\lambda < 1$ so that $1/\lambda > 1$, for each n

$$\begin{aligned} h_r^{-1}{}_{k\in I_r} z_{k,n}^{\lambda} &= {}_{k\in I_r} \left(h_r^{-1} z_{k,n} \right)^{\lambda} \left(h_r^{-1} \right)^{1-\lambda} \\ &\leq \left({}_{k\in I_r} \left[\left(h_r^{-1} z_{k,n} \right)^{\lambda} \right]^{1/\lambda} \right)^{\lambda} \left({}_{k\in I_r} \left[\left(h_r^{-1} \right)^{1-\lambda} \right]^{1/(1-\lambda)} \right)^{1-\lambda} \\ &= \left(h_r^{-1}{}_{k\in I_r} z_{k,n} \right)^{\lambda} \end{aligned}$$

by Hölder's inequality, and thus

$$h_r^{-1}{}_{k\in I_r} \mu_{k,n}^{\lambda_k} \leq h_r^{-1}{}_{k\in I_r} \mu_{k,n} + \left[h_r^{-1}{}_{k\in I_r} z_{k,n} \right]^{\lambda}.$$

Hence $x \in [w^{\theta}, \mathcal{M}, p]_{\sigma}^{\infty}(\Delta_v^m)$. (ii) and (iii) can be proved similar to (i).

Theorem 2.13 ***(i)*** *The sequence spaces* $[w^{\theta}, \mathcal{M}, p, u]_{\sigma}^{\infty}$ *and* $[w^{\theta}, \mathcal{M}, p, u]_{\sigma}^{0}$ *are solid and hence are monotone.*

(ii) *The space* $[w^{\theta}, \mathcal{M}, p, u]_{\sigma}$ *is not monotone and a such is neither solid nor perfect.*

Proof. (i) We give the proof for $[w^{\theta}, \mathcal{M}, p, u]_{\sigma}^{0}$. Let $x \in [w^{\theta}, \mathcal{M}, p, u]_{\sigma}^{0}$ and (α_k) be sequences of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$h_r^{-1}{}_{k\in I_r} u_k \left[M_k \left(\frac{|t_{kn}(\alpha_k x_k)|}{\rho} \right) \right]^{p_k} \leq h_r^{-1}{}_{k\in I_r} u_k \left[M_k \left(\frac{|t_{kn}(x_k)|}{\rho} \right) \right]^{p_k} \to 0$$

$(r \to \infty)$, uniformly in n. Hence $\alpha x \in [w^{\theta}, \mathcal{M}, p, u]_{\sigma}^{0}$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$, whenever $x \in [w^{\theta}, \mathcal{M}, p, u]_{\sigma}^{0}$. The spaces are monotone follows from the Remark 1.

Theorem 2.14 *Let* $\theta = (k_r)$ *be a lacunary sequence. If* $1 < \liminf_r q_r < \limsup_r q_r < \infty$. *Then* $[w, \mathcal{M}, p, u]_{\sigma}(\Delta_v^m) = [w^{\theta}, \mathcal{M}, p, u]_{\sigma}(\Delta_v^m)$, *where*

$$[w, \mathcal{M}, p, u]_{\sigma}(\Delta_v^m) = \left\{ x = (x_k) : \lim_i \frac{1}{i} {}_{k=1}^{i} u_k \left[M_k \left(\frac{|t_{kn}(\Delta_v^m x_k - le)|}{\rho} \right) \right]^{p_k} = 0 \right\}$$

for some l, $\rho > 0$ *and uniformly in* n.

Proof. Suppose that $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = (\frac{k_r}{k_{r-1}}) \geq 1 + \delta$ for all $r \geq 1$. Furthermore we have $\frac{k_r}{h_r} \leq \frac{(1+\delta)}{\delta}$ and $\frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}$, for all $r \geq 1$. Then we may write

$$\frac{1}{h_r}{}_{j\in I_r} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}$$

$$\begin{aligned}
&= \frac{1}{h_r}{}_{j=1}^{k_r} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j} - \frac{1}{h_r}{}_{j=1}^{k_{r-1}} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j} \\
&= \frac{k_r}{h_r}\left(k_r^{-1}{}_{j=1}^{k_r} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}\right) \\
&\quad -\frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}{}_{j=1}^{k_{r-1}} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}\right).
\end{aligned}$$

Now suppose that $\limsup_r q_r < \infty$ and let $\varepsilon > 0$ be given. Then there exists s_0 such that for every $s \geq s_0$

$$A_s = \frac{1}{h_s}{}_{k\in I_s} u_k \left[M_k\left(\frac{|t_{kn}(\Delta_v^m x_k)|}{\rho}\right)\right]^{p_k} < \varepsilon.$$

We can also choose a number $K > 0$ such that $A_s < K$ for all s. If $\limsup_r q_r < \infty$, then there exists a number $\beta > 0$ such that $q_r < \beta$ for all r. Now let i be any integer with $k_{r-1} < i \leq k_r$, where $r > L$. Then

$$i^{-1}{}_{j=1}^{i} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}$$

$$\begin{aligned}
&\leq k_{r-1}^{-1}{}_{j=1}^{k_r} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j} \\
&= k_{r-1}^{-1}\left\{{}_{j\in I_1} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j} + {}_{j\in I_2} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}\right. \\
&\quad \left. +\ldots + {}_{j\in I_r} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}\right\}
\end{aligned}$$

$$\begin{aligned}
&= k_{r-1}^{-1}\left\{{}_{s=1}^{s_0}{}_{j\in I_s} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j} + {}_{s=s_0+1}^{r}{}_{j\in I_s} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j}\right\} \\
&= k_{r-1}^{-1}\left\{{}_{s=1}^{s_0}{}_{j\in I_s} u_j \left[M_j\left(\frac{|t_{jn}(\Delta_v^m x_j)|}{\rho}\right)\right]^{p_j} + \varepsilon(k_r - k_{s_0})k_{r-1}^{-1}\right\} \\
&\leq k_{r-1}^{-1}\{h_1 A_1 + h_2 A_2 + \ldots + h_{s_0} A_{s_0}\} + \varepsilon(k_r - k_{s_0})k_{r-1}^{-1} \\
&\leq k_{r-1}^{-1}\left(\sup_{1\leq j\leq s_0} A_s\right) k_{s_0} + \varepsilon(k_r - k_{s_0})k_{r-1}^{-1} \\
&< K k_{r-1}^{-1} k_{s_0} + \varepsilon\beta
\end{aligned}$$

which yields that $x \in [w, \mathcal{M}, p, u]_\sigma(\Delta_v^m)$.

3 Open Problem

The aim of this paper is to introduce and study the new sequence spaces $[w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m)$, $[w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m)$ and $[w^\theta, \mathcal{M}, p, u]_\sigma^0(\Delta_v^m)$. We propose to study various some topological properties and establish some inclusion relations between these spaces. But we didn't prove inclusion relations $[w^\theta, \mathcal{M}, p, u]_\sigma^0$ $(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p, u]_\sigma(\Delta_v^m) \subset [w^\theta, \mathcal{M}, p, u]_\sigma^\infty(\Delta_v^m)$. Therefore it is left as an open problem.

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