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Some Certain Differantial Identities in Semiprime Rings With Bi-Derivation

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Abstract

In ring theory, examining the properties of derivations is an important field of study. This field is expanding as researchers define different derivations and find new conditions. In this paper, we investigated the concept of symmetric bi-derivations of a nonzero two-sided ideal of a ring and analyzed some related properties. Throughout the work, we aimed to obtain more general results by taking a nonzero two-sided ideal of a semiprime ring instead of a ring. In addition, we adapted some conditions that have been studied for many years to bi-derivations of a nonzero two-sided ideal of the semiprime ring and obtained new results.

Keywords: *ring, semiprime ring, ideal, derivation, bi-derivation, symmetric bi-derivation.*
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1 Introduction

Throughout this paper, S stands for an associative ring characteristic different from 2 and $Z(S)$ the center of S. A ring S is said to be prime if $a_1Sa_2 = (0)$ implies that either $a_1 = 0$ or $a_2 = 0$ and semiprime if $a_1Sa_1 = (0)$.implies that $a_1 = 0$, where $a_1, a_2 \in S$. A prime ring is obviously semiprime ring. A derivation d is inner if there exists a fixed $a \in S$ such that $d(a_1) = [a, a_1]$

holds for all $a_1 \in S$. For any $a_1, a_2 \in S$, as usual $[a_1, a_2] = a_1a_2 - a_2a_1$ and $a_1oa_2 = a_1a_2 + a_2a_1$ will denote the well-known Lie and Jordan products respectively.. An additive mapping $d : S \to S$ is said to be a derivation if $d(a_1a_2) = a_1d(a_2) + d(a_1)a_2$ for all $a_1, a_2 \in S$.

A mapping $D(.,.) : S \times S \to S$ is said to be symmetric if $D(a_1, a_2) = D(a_2, a_1)$ for all $a_1, a_2 \in S$. A symmetric bi-additive mapping $D(.,.) : S \times S \to S$ is called a symmetric bi-derivation if $D(a_1a_2, a_3) = D(a_1, a_3)a_2 + a_1D(a_2, a_3)$ for all $a_1, a_2, a_3 \in S$. Obviously, in this case also the relation $D(a_1, a_2a_3) = D(a_1, a_2)a_3 + a_2D(a_1, a_2)$ for all $a_1, a_2, a_3 \in S$ holds. A mapping $d : S \to S$ is called the trace of $D(.,.)$ if $d(a_1) = D(a_1, a_1)$ for all $a_1 \in S$. It is obvious that $D(.,.)$ is bi-additive (i.e. additive in both arguments), then trace d of $D(.,.)$ satisfies the identity $d(a_1 + a_2) = d(a_1) + d(a_2) + 2D(a_1, a_2)$for all $a_1, a_2 \in S$. To best of my knowladge, the concept of symmetric bi-derivation appeared for the first time in the work of G. Maksa [2] (see also [1] where an example can be found). It is shown in [2] that symetric bi-derivations are related to general solution of some functional equations. In [3] ,Vukman examined symmetric bi-derivations on rings with centralizing mappings and obtained at some notable results concerning symmetric bi-derivations on prime rings. Inspired by this study, Yenigül and Argaç [4] obtained important results regarding symmetric bi-derivations for nonzero ideals of rings. Later, concept of generalized symmetric bi-derivations was introduced in [6] by Argaç. Recently, in [7], Shujat was proved some results concerning symmetric generalized derivations on prime and semiprime rings.

The present paper aims to extend some well-known results regarding symmetric bi-derivations in rings to two-sided ideals of semiprime rings. Thus, it is to be contributed to the theory of centralizing and commuting symmetric bi-derivations in an ideal of semiprime rings.

In this paper, we shall frequently use the following identities and several well known facts about the semiprime rings without specific mention.

$[a_1a_2, a_3] = a_1[a_2, a_3] + [a_1, a_3]a_2$ and $[a_1, a_2a_3] = a_2[a_1, a_3] + [a_1, a_2]a_3$
$a_1o(a_2a_3) = (a_1oa_2)a_3 - a_2[a_1, a_3] = a_2(a_1oa_3) + [a_1, a_2]a_3$
$(a_1a_2)oa_3 = a_1(a_2oa_3) - [a_1, a_3]a_2 = (a_1oa_3)a_2 + a_1[a_2, a_3]$.

2 Results

Lemma 2.1 *[5, Lemma 2.1] Let S be a semiprime ring, I a nonzero two-sided ideal of S and $a \in S$ such that $aa_1a = 0$ for all $a_1 \in I$, then $a = 0$.*

Theorem 2.2 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D(I, I) \subseteq Z(I)$, then $[d(a_1), a_1] = 0$, for all $a_1 \in I$.*

Proof: We have.

$$[D(a_1, a_2), a_3] = 0 \text{ for all } a_1, a_2, a_3 \in I. \tag{1}$$

Replacing a_1 by $a_1 t$ in (1), we obtain that

$$\begin{aligned} 0 &= [D(a_1 t, a_2), a_3] \\ &= [a_1 D(t, a_2) + D(a_1, a_2) t, a_3] \end{aligned}$$

and so

$$a_1 [D(t, a_2), a_3] + [a_1, a_3] D(t, a_2) + [D(a_1, a_2), a_3] t + D(a_1, a_2) [t, a_3] = 0 \text{ for all } a_1, a_2, a_3, t \in I.$$

Using (1), we get

$$[a_1, a_3] D(t, a_2) + D(a_1, a_2) [t, a_3] = 0 \text{ for all } a_1, a_2, a_3, t \in I. \tag{2}$$

Now substituting a_1 instead of a_3 in (2), we find that

$$D(a_1, a_2) [t, a_1] = 0 \text{ for all } a_1, a_2, t \in I. \tag{3}$$

Replacing $a_1 a_2$ instead of a_2 in (3) and using this, we get

$$0 = D(a_1, a_1 a_2) [t, a_1] = a_1 D(a_1, a_2) [t, a_1] + D(a_1, a_1) a_2 [t, a_1]$$

and so

$$d(a_1) a_2 [t, a_1] = 0 \text{ for all } a_1, a_2, t \in I. \tag{4}$$

Taking $a_1 a_2$ by a_2 in this expression, we have

$$d(a_1) a_1 a_2 [t, a_1] = 0 \text{ for all } a_1, a_2, t \in I. \tag{5}$$

Multipliying equation (4) on the left by a_1, we obtain that

$$a_1 d(a_1) a_2 [t, a_1] = 0 \text{ for all } a_1, a_2, t \in I. \tag{6}$$

Subtracting (5) from (6), we arrive at

$$[d(a_1), a_1] a_2 [t, a_1] = 0 \text{ for all } a_1, a_3, t \in I.$$

Taking t by $d(a_1)$ in last equation, we get

$$[d(a_1), a_1] I [d(a_1), a_1] = (0) \text{ for all } a_1 \in I.$$

By Lemma 2.1, we obtain that $[d(a_1), a_1] = 0$ for all $a_1 \in I$.

Theorem 2.3 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D(a_1, a_2)\, oa_3 = 0$ for all $a_1, a_2, a_3 \in I$, then $[d(a_1), a_1] = 0$, for all $a_1 \in I$.*

Proof: Given

$$D(a_1, a_2)\, oa_3 = 0 \text{ for all } a_1, a_2, a_3 \in I. \tag{7}$$

Replacing $a_1 t$ instead of a_1 in (7), we get

$$D(a_1 t, a_2)\, oa_3 = 0 \text{ for all } a_1, a_2, a_3, t \in I.$$

Expanding this, we have

$$\begin{aligned} 0 &= (a_1 D(t, a_2) + D(a_1, a_2)\, t)\, oa_3 \\ &= a_1 (D(t, a_2)\, oa_3) - [a_1, a_3]\, D(t, a_2) + (D(a_1, a_2)\, oa_3)\, t + D(a_1, a_2)\, [t, a_3] \end{aligned}$$

using the hypothesis in this equation, we obatin

$$D(a_1, a_2)\, [t, a_3] = [a_1, a_3]\, D(t, a_2) \text{ for all } a_1, a_2, a_3, t \in I. \tag{8}$$

Writting a_1 instead of a_3, a_2 in (8), we arrive at

$$d(a_1)\, [t, a_1] = 0 \text{ for all } a_1, t \in I. \tag{9}$$

Taking t by $tr, r \in S$ in (9) and using (9)

$$\begin{aligned} 0 &= d(a_1)\, [tr, a_1] \\ &= d(a_1)\, t\, [r, a_1] + d(a_1)\, [t, a_1]\, r \end{aligned}$$

and so

$$d(a_1)\, t\, [r, a_1] = 0 \text{ for all } a_1, t \in I, r \in S. \tag{10}$$

Replacing $a_1 t$ instead of t in (10), we get

$$d(a_1)\, a_1 t\, [r, a_1] = 0 \text{ for all } a_1, t \in I, r \in S. \tag{11}$$

On the other hand, multipliying equation (10) on the left for a_1, we have

$$a_1 d(a_1)\, t\, [r, a_1] = 0 \text{ for all } a_1, t \in I, r \in S. \tag{12}$$

By comparing (10) and (11) equalities, we find that

$$[d(a_1), a_1]\, t\, [r, a_1] = 0 \text{ for all } a_1, t \in I, r \in S.$$

Writing r instead of $d(a_1)$ in last equation, we get

$$[d(a_1), a_1]\, t\, [d(a_1), a_1] = 0 \text{ for all } a_1, t \in I.$$

It follows that

$$[d(a_1), a_1]\, S\, [d(a_1), a_1] = 0 \text{ for all } a_1 \in I.$$

Finally, using Lemma 2.1, we arrive at $[d(a_1), a_1] = 0$ for all $a_1 \in I$.

Theorem 2.4 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $[D(a_1,a_2),a_3]=[a_1,a_3]$ for all $a_1,a_2,a_3\in I$, then $[d(a_1),a_1]=0$, for all $a_1\in I$.*

Proof: We have

$$[D(a_1,a_2),a_3]=[a_1,a_3] \text{ for all } a_1,a_2,a_3\in I. \tag{13}$$

Taking a_1 by a_1t in (13), we get

$$[D(a_1t,a_2),a_3]=[a_1t,a_3] \text{ for all } a_1,a_2,a_3,t\in I.$$

Expanding this equation and using (13), we have

$$a_1[t,a_3]+[a_1,a_3]t=a_1[D(t,a_2),a_3]+[a_1,a_3]D(t,a_2)+[D(a_1,a_2),a_3]t+D(a_1,a_2)[t,a_3]$$

and so

$$[a_1,a_3]D(t,a_2)+D(a_1,a_2)[t,a_3]=0 \text{ for all } a_1,a_2,a_3,t\in I. \tag{14}$$

Replacing a_1 instead of a_3,a_2 in (14), we obtain

$$d(a_1)[t,a_1]=0 \text{ for all } a_1,t\in I.$$

This is the same as equation (9). Using the same arguments as we used in the proof of Theorem 2.3, we get the required result.

Theorem 2.5 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D(a_1,a_2)oa_3=[a_1,a_3]$ for all $a_1,a_2,a_3\in I$, then $[d(a_1),a_1]=0$, for all $a_1\in I$.*

Proof: We have

$$D(a_1,a_2)oa_3=[a_1,a_3] \text{ for all } a_1,a_2,a_3\in I. \tag{15}$$

Writting a_1t instead of a_1 in (15) and using (15), we get

$$D(a_1t,a_2)oa_3=[a_1t,a_3]$$

$$a_1[t,a_3]+[a_1,a_3]t=a_1(D(t,a_2)oa_3)-[a_1,a_3]D(t,a_2)+(D(a_1,a_2)oa_3)t+D(a_1,a_2)[t,a_3]$$

$$D(a_1,a_2)[t,a_3]=[a_1,a_3]D(t,a_2) \text{ for all } a_1,a_2,t\in I. \tag{16}$$

Replacing a_1 instead of a_3,a_2 in (16), we find that

$$d(a_1)[t,a_1]=0 \text{ for all } a_1,t\in I.$$

Using the same arguments after (9) in the proof of Theorem 2.3, we get the required result.

Theorem 2.6 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D([a_1, a_2], a_3) = 0$ for all $a_1, a_2, a_3 \in I$, then $[d(a_1), a_1] = 0$, for all $a_1 \in I$.*

Proof: By the hypothesis, we have

$$D([a_1, a_2], a_3) = 0 \text{ for all } a_1, a_2, a_3 \in I. \tag{17}$$

Replacing $a_1 t$ by a_1 in (17), we get

$$\begin{aligned} 0 &= D([a_1 t, a_2], a_3) \\ &= D(a_1 [t, a_2] + [a_1, a_2] t, a_3) \\ &= a_1 D([t, a_2], a_3) + D(a_1, a_3)[t, a_2] + [a_1, a_2] D(t, a_3) + D([a_1, a_2], a_3) t \end{aligned}$$

By hypothesis, we obtain that

$$D(a_1, a_3)[t, a_2] + [a_1, a_2] D(t, a_3) = 0 \text{ for all } a_1, a_2, a_3, t \in I. \tag{18}$$

Taking a_1 instead of a_3, a_2 in (18), we find that

$$d(a_1)[t, a_1] = 0 \text{ for all } a_1, t \in I.$$

Using the same arguments after (9) in the proof of Theorem 2.3, we get the required result.

Theorem 2.7 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D([a_1, a_2], a_3) = [a_1, a_2]$ for all $a_1, a_2, a_3 \in I$, then $[d(a_1), a_1] = 0$, for all $a_1 \in I$.*

Proof: By the hypothesis, we have

$$D([a_1, a_2], a_3) = [a_1, a_2] \text{ for all } a_1, a_2, a_3 \in I. \tag{19}$$

Substituting $a_1 t$ by a_1 in (19), we get

$$[a_1 t, a_2] = D([a_1 t, a_2], a_3) = D(a_1 [t, a_2] + [a_1, a_2] t, a_3)$$
$$a_1 [t, a_2] + [a_1, a_2] t = a_1 D([t, a_2], a_3) + D(a_1, a_3)[t, a_2] + [a_1, a_2] D(t, a_3) + D([a_1, a_2], a_3) t$$

Using (19) in last equation, we obtain that

$$D(a_1, a_3)[t, a_2] + [a_1, a_2] D(t, a_3) \text{ for all } a_1, a_2, a_3, t \in I. \tag{20}$$

Putting a_1 instead of a_3, a_2 in (20), we find that

$$d(a_1)[t, a_1] = 0 \text{ for all } a_1, t \in I.$$

Using the same arguments after (9) in the proof of Theorem 2.3, we get the required result.

The result of the following theorem is easily achieved by the methods used in the proof of Theorem 2.7.

Theorem 2.8 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D([a_1,a_2],a_3) = [a_1,a_3]$ for all $a_1,a_2,a_3 \in I$, then $[d(a_1),a_1] = 0$, for all $a_1 \in I$.*

Theorem 2.9 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D(a_1oa_2,a_3) = 0$ for all $a_1,a_2,a_3 \in I$, then $[d(a_1),a_1] = 0$, for all $a_1 \in I$.*

Proof: Given

$$D(a_1oa_2,a_3) = 0 \text{ for all } a_1,a_2,a_3 \in I. \tag{21}$$

which means that

$$D(a_1,a_3)oa_2 + a_1oD(a_2,a_3) = 0 \text{ for all } a_1,a_2,a_3 \in I. \tag{22}$$

Writing a_2a_1 instead of a_2 in (22) and using (22), we have

$$\begin{aligned} 0 &= D(a_1,a_3)oa_2a_1 + a_1oD(a_2a_1,a_3) \\ &= (D(a_1,a_3)oa_2)a_1 - a_2[D(a_1,a_3),a_1] + a_1o(a_2D(a_1,a_3) + D(a_2,a_3)a_1) \\ &= (D(a_1,a_3)oa_2)a_1 - a_2[D(a_1,a_3),a_1] + (a_1oa_2)D(a_1,a_3) - a_2[a_1,D(a_1,a_3)] + (a_1oD(a_2,a_3)) \\ &= (D(a_1,a_3)oa_2 + a_1oD(a_2,a_3))a_1 + (a_1oa_2)D(a_1,a_3) \end{aligned}$$

and so

$$(a_1oa_2)D(a_1,a_3) = 0 \text{ for all } a_1,a_2,a_3 \in I. \tag{23}$$

Taking a_2 by $ra_2, r \in S$ and a_3 by a_1 in (23) and using (23), we obtain that

$$[a_1,r]a_2d(a_1) = 0 \text{ for all } a_1,a_2,a_3 \in I, r \in S.$$

This is the same as equation (10). Using the same arguments as we used in the proof of Theorem 2.3, we get the required result.

Theorem 2.10 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D(a_1oa_2,a_3) = a_1oa_2$ for all $a_1,a_2,a_3 \in I$, then $[d(a_1),a_1] = 0$, for all $a_1 \in I$.*

Proof: By the hypothesis, we have

$$D(a_1oa_2,a_3) = a_1oa_2 \text{ for all } a_1,a_2,a_3 \in I. \tag{24}$$

This expression is also equal to the following

$$D(a_1,a_3)oa_2 + a_1oD(a_2,a_3) = a_1oa_2 \text{ for all } a_1,a_2,a_3 \in I. \tag{25}$$

Replacing a_2a_1 by a_2 in (25) and using (25), we find that

$$(a_1oa_2)\, D\,(a_1, a_3) = 0 \text{ for all } a_1, a_2, a_3 \in I.$$

After this equation, using the same arguments as we used in the proof of Theorem 2.9, we get the required result.

Theorem 2.11 *Let S be a $2-$torsion free semiprime ring, I a nonzero two-sided ideal of S. Suppose that S admits a D a symmetric bi-derivation and d be the trace of D. If $D\,(a_1oa_2, a_3) = [a_1, a_2]$ for all $a_1, a_2, a_3 \in I$, then $[d(a_1), a_1] = 0$, for all $a_1 \in I$.*

Proof:
We have

$$D\,(a_1oa_2, a_3) = [a_1, a_2] \text{ for all } a_1, a_2, a_3 \in I. \tag{26}$$

Expanding this equation, we get

$$D\,(a_1, a_3)\, oa_2 + a_1oD\,(a_2, a_3) = [a_1, a_2] \text{ for all } a_1, a_2, a_3 \in I. \tag{27}$$

Substituting a_2a_1 by a_2 in (27) and using (27), we obtain that

$$(a_1oa_2)\, D\,(a_1, a_3) = 0 \text{ for all } a_1, a_2, a_3 \in I.$$

After last equation, using the same arguments as we used in the proof of Theorem 2.9, we get the required result.

3 Open Problem

In this study, some properties of a nonzero two-sided ideal of a semiprime ring with bi-derivation have been shown. Moreover, some well-known results in derivation have been adapted to symmetric bi-derivation. The findings obtained in there are can help examine properties of symmetric bi-derivation in Lie ideals or square-closed Lie ideals of rings. Besides, these results we obtained can be investigated for the symmetric generalized bi-derivation.

References

[1] Maksa, G. (1980). A remark on symmetric bi-additive functions having non-negative diagonalization. Glasnic Math. 15: 279-280.

[2] Maksa, G. (1987). On trace of symmetric bi-derivation. C. R. Math. Rep. Acad. Sci. Canada. 9: 303-307.

[3] Vukman, J. (1989). Symmetric bi-derivations on prime and semiprime rings. Aequations Math. 38: 245-254.

[4] Yenigul, M. S, Argac, N. (1993). Ideals and symmetric biderivations of prime and semi-prime rings. Math. J. Okayama Univ., 35:189-192.

[5] Samman, M. S., Thaheem, A. B. (2003). Derivations on semiprime rings. Int. J. Pure Appl. Math. 5(4):465–472.

[6] Argac N., On prime and semiprime rings with derivations, Algebra Colloquium, 13 (3) (2006) 371-380.

[7] Shujat F., Symmetric generalized biderivations on prime rings, Boletim da Socieade Paranaense de Matematica, 39 (4) (2021) 65-72.