Int. J. Open Problems Compt. Math., Vol. 17, No. 4, December 2024
Print ISSN: 1998-6262, Online ISSN: 2079-0376

On a Class of Differential Equations Involving Several Sequential Caputo Derivatives

S. Chibane[1], Y. Gouari[2], Z. Dahmani[3], M. Kaid[4]

[1,2]Higher Normal School of Mostaganem, Abdelhamid Bni Badis University, Mostaganem, Algeria.

[3]Laboratory LAMDA-RO, University of Blida 1, Algeria.

[4]Laboratory of Pure and Applied Mathematics, Abdelhamid Bni Badis University, Mostaganem, Algeria.

e-mail: [1]chibane028@gmail.com, [2]gouariyazid@gmail.com, [3]zzdahmani@yahoo.fr,[4]mohammed.kaid@univ-mosta.dz

Received 20 November 2024, Accepted 28 December 2024

Abstract

This paper presents a first main result on the existence and uniqueness of solutions for a class of sixth order nonlinear fractional differential equations involving six sequential Caputo derivatives. A second main result on Ulam- Hyers stability of solutions is also discussed. At the end, two examples are discussed to show the applicability of the main findings.

Keywords: *Six Sequential Caputo derivatives, Existence and uniqueness, Fixed point, Ulam-Hyers stability.*

2010 Mathematics Subject Classification: 34A08, 26A33.

1 Introduction

Fractional calculus is a branch of mathematics that generalizes derivatives and integrals to non-integer orders, allowing them to take any real or complex value. This generalization provides a powerful tool for modeling processes that exhibit nonlocal or memory-dependent behavior, which are often encountered in fields such as physics, biology, engineering, and finance. Unlike classical calculus, fractional calculus naturally captures systems where the current state depends on the entire history of the process, making it very suitable for studying phenomena like anomalous diffusion, viscoelastic materials, and control systems, see, for example [3,6,17]. The Caputo derivative is particularly useful for initial-value problems because it incorporates traditional initial conditions. However, a key issue in the Caputo and Riemann-Liouville definitions is the singularity of their non-local kernels, which limits their applicability to real-world problems, as outlined in [4,12,25,30,32,36]. The present paper is motivated by the need to cite and recall some classes of nonlinear differential equations of high order, specifically in the context of the standard derivatives with order six since the all the studied problem of sixth order can be seen as limiting cases of our problem. These equations, accompanied by boundary boundary conditions, provide a natural setting for incorporating global information about the solution. Such problems arise in various applications, see for instance [5,8,10,20,26,27,34,37]. R.P. Agarwal et al. in [2], the authors have studied the equilibrium state of an elastic circular ring segment with its two ends by a following problem of sixth-order:

$$\begin{cases} v^{(6)} + 2v^{(4)} + v'' = f\left(x, v\right), & \text{in } \Omega = (0,1)\,, \\ v = v'' = v^{(4)} = 0, & \text{on } \partial\Omega. \end{cases}$$

In [28], the authors studied the existence of positive solutions of the nonlinear boundary value problem:

$$\begin{cases} w^{(6)} + f\left(x, w, w'', w^{(4)}\right) = 0, & \text{in } \Omega = (0,1)\,, \\ w = w'' = w^{(4)} = 0, & \text{on } \partial\Omega. \end{cases}$$

Then in [15], C.P. Danet studied the existence and multiplicity of solutions for the problem:

$$\begin{cases} v^{(6)} + Av^{(4)} + Bv'' + C\left(x\right)v + f\left(x, v\right) = 0, & \text{in } \Omega = (0,1)\,, \\ v = v'' = v^{(4)} = 0, & \text{on } \partial\Omega. \end{cases}$$

In the article [11], the author investigated the existence, regularity, and uniqueness of solutions for the boundary value problem associated with a sixth-order partial differential equation. He used classical methods, such as the maximum principle and the method of P-functions, and extended uniqueness results for equations with non-constant coefficients in higher dimensions.

$$\Delta^3 v - \widetilde{B}(t)\Delta^2 v + \widetilde{C}(t)\Delta v - \widetilde{D}(t)v = G(t, v), \quad \text{in } \Omega,$$

with boundary conditions $v = \Delta v = \Delta^2 v = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{N}^M$ is a bounded domain.

In a very recent paper, M. Kaid et al. [23], discussed the existence and uniqueness of solutions for the following class of alpha-fractional order

$$\begin{cases} {}^CD^{\alpha}v(t) = M_1F_1(t, v^{(4)}(t)) + M_2F_2(t, v^{(2)}(t)) + M_3F_3(t, v(t)), \quad t \in [0,1], \\ v(0) = a_0, \quad v(1) = b_0, \quad a_0,\ b_0 \in \mathbb{R}, \\ v'(0) = a_1, \quad v'(1) = b_1, \quad a_1,\ b_1 \in \mathbb{R}, \\ v''(0) = a_2, \quad v''(1) = b_2, \quad a_2,\ b_2 \in \mathbb{R}, \end{cases}$$

where ${}^CD^{\alpha}$ denote the Caputo fractional derivatives of order α such that $5 < \alpha \leq 6$, and $u^{(\eta)}(t)$, $\eta \in \{0, 2, 4\}$ is derivative of the function u with respect to t, where v: $[0, 1] \to \mathbb{R}$ is a given continuous function and M_l $(l = \overline{1, 3})$ are given constants.

Very recently, Bezziou et al. [7], discussed the existence and uniqueness of solutions under boundary conditions of the form:

$$\begin{cases} {}^C_HD^{\delta}u(t) = \lambda_1G_1(t, u^{(4)}(t)) + \lambda_2G_2(t, u^{(2)}(t)) + \lambda_3G_3(t, u(t)), \quad t \in [1,e], \\ u(1) = u'(1) = u''(1) = 0, \\ u(e) = u'(e) = 0, u''(e) = \rho \int_1^e u(t)\dfrac{dt}{t}, \quad (\lambda_k)_{k=\overline{1,3}}, \rho \in \mathbb{R}, \end{cases}$$

where ${}^C_HD^{\delta}$ denote the Caputo-Hadamard fractional derivative of order δ, such that $5 < \delta \leq 6$, and $u^{(\gamma)}(t)$, $\gamma \in \{0, 2, 4\}$ are derivatives of u with respect to t.

Also in [19], the authors discussed the existence and uniqueness for the following equation:

$$\begin{aligned} D^{\alpha}D^{\beta}D^{\gamma}\chi(t) &= F_1(t, \chi(t), D^{\beta}D^{\gamma}\chi(t)) + F_2(t, \chi(t), D^{\gamma}\chi(t)) \\ &\quad + F_3(t, \chi(t), I^{\rho}\chi(t)) + F_4(t, \chi(t)), \quad t \in [0,1], \end{aligned}$$

with boundary conditions:

$$\begin{cases} \chi(0) = \theta_0, \quad \chi(1) = \mu_0, \\ D^{\gamma}\chi(t)(0) = \theta_1, \quad D^{\gamma}\chi(1) = \mu_1, \\ D^{\beta}D^{\gamma}\chi(0) = \theta_2, \quad D^{\beta}D^{\gamma}\chi(1) = \mu_2, \quad \theta_i,\ \mu_i \in \mathbb{R},\ i = \overline{0, 2}, \end{cases}$$

where ${}^CD^{\alpha}, {}^CD^{\beta}, {}^CD^{\gamma}$ denote the Caputo fractional derivatives of order α, β, γ such that $1 < \alpha, \beta, \gamma \leq 2$, I^{ρ} is the Riemann-Liouville integral of order ρ such that $\rho > 0$. the function χ with respect to t, where χ: $[0, 1] \to \mathbb{R}$ is given continuous function.

In this work, we discuss the existence and uniqueness of solutions for the following sequential fractional differential problem:

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t) = m_1(t,x(t),D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t)) + m_2(t,x(t),D^{\alpha_5}D^{\alpha_6}x(t)) \\ +m_3(t,x(t),I^{\xi}x(t)) + m_4(t,x(t)), \quad t\in[0,1], \\ x(0)=\epsilon_0, \quad x(1)=\lambda_0, \quad \epsilon_0,\ \lambda_0\in\mathbb{R}, \\ D^{\alpha_6}x(0)=\epsilon_1, \quad D^{\alpha_6}x(1)=\lambda_1, \quad \epsilon_1,\ \lambda_1\in\mathbb{R}, \\ D^{\alpha_5}D^{\alpha_6}x(0)=\epsilon_2, \quad D^{\alpha_5}D^{\alpha_6}x(1)=\lambda_2, \quad \epsilon_2,\ \lambda_2\in\mathbb{R}, \end{cases} \tag{1}$$

where D^{α_i} denote the Caputo fractional derivatives of order α_i such that $0 < \alpha_i \leq 1$, and $I^{\xi}x$ is the Riemann-Liouville fractional integral order $\xi \in [0,+\infty[$.

2 Preliminaries on Caputo Derivatives

We need to introduce the Caputo derivatives. For more details, we refer to the references [1,9,13,14,18,22,23,30].

Definition 2.1 *Let $\alpha > 0$ and $f : J \longmapsto \mathbb{R}$ be a continuous function. The Riemann-Liouville integral is defined by:*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau.$$

Definition 2.2 *For any $f \in C^n(J,\mathbb{R})$ and $n-1 < \alpha \leq n$, the Caputo derivative is defined by:*

$$\begin{aligned} D^{\alpha}f(t) &= I^{n-\alpha}\frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds. \end{aligned}$$

To study (1), we need the following two results [?]:

Lemma 2.3 *Let $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$. Then, the general solution of $D^{\alpha}y(t) = 0; t \in J$ is:*

$$y(t) = \sum_{i=0}^{n-1} c_i t^i,$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,..,n-1$.

Lemma 2.4 *If $n \in \mathbb{N}^*$, and $n-1 < \alpha < n$, then, we have*

$$I^{\alpha}D^{\alpha}y(t) = y(t) + \sum_{i=0}^{n-1} c_i t^i,$$

and $c_i \in \mathbb{R}$, $i = 0,1,2,..,n-1$.

Now, let us prove the following integral equation.

Lemma 2.5 *Let $S \in C(I)$. Then,*

$$\begin{cases} D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}\ x(t) = S(t), \quad t \in [0,1], \\ x(0) = \epsilon_0, \qquad x(1) = \lambda_0, \qquad \epsilon_0,\ \lambda_0 \in \mathbb{R}, \\ D^{\alpha_6}\ x(t)(0) = \epsilon_1, \qquad D^{\alpha_6}\ x(1) = \lambda_1, \qquad \epsilon_1,\ \lambda_1 \in \mathbb{R}, \\ D^{\alpha_5}D^{\alpha_6}\ x(0) = \epsilon_2, \qquad D^{\alpha_5}D^{\alpha_6}\ x(1) = \lambda_2, \qquad \epsilon_2,\ \lambda_2 \in \mathbb{R}, \end{cases}$$

if and only if

$$\begin{aligned} x(t) &= I^{\sum_{i=1}^{6}\alpha_i} S(t) + (\Gamma_2^4)^{-1}\Gamma_2^6\Big[\varphi_1 - \varphi_2 I^{\sum_{i=1}^{4}\alpha_i} S(1) + \varphi_3 I^{\sum_{i=1}^{5}\alpha_i} S(1) + \varphi_4 I^{\sum_{i=1}^{6}\alpha_i} S(1)\Big] \\ &\quad \times t^{\sum_{i=2}^{6}\alpha_i} + \Gamma_3^6\Big[\phi_1 + \phi_2 I^{\sum_{i=1}^{4}\alpha_i} S(1) + \phi_3 I^{\sum_{i=1}^{6}\alpha_i} S(1) - \frac{G_1G_2}{\delta} I^{\sum_{i=1}^{5}\alpha_i} S(1)\Big] t^{\sum_{i=3}^{6}\alpha_i} \\ &\quad + \Gamma_4^6\Big[\Delta_1 + \Delta_2 I^{\sum_{i=1}^{4}\alpha_i} S(1) + \frac{G_1}{\delta} I^{\sum_{i=1}^{5}\alpha_i} S(1) - \frac{K_1}{\delta} I^{\sum_{i=1}^{6}\alpha_i} S(1)\Big] t^{\sum_{i=4}^{6}\alpha_i} \\ &\quad + \epsilon_2\Gamma_5^6 t^{\sum_{i=5}^{6}\alpha_i} + \epsilon_1\Gamma_6^6 t^{\alpha_6} + \epsilon_0 \end{aligned} \tag{2}$$

where $\delta \neq 0$ and

$$\varphi_1 = Z_3 - \Gamma_3^4\phi_1 - \Gamma_4^4\Delta_1,\ \varphi_2 = 1 + \Gamma_3^4\phi_2 + \Gamma_4^4\Delta_1,\ \varphi_3 = \Gamma_3^4\frac{G_1G_2}{\delta} - \Gamma_4^4\frac{G_1}{\delta},\ \varphi_4 = \Gamma_4^4\frac{K_1}{\delta} - \Gamma_3^4\phi_3$$

$$\phi_1 = (G_1)^{-1}G_3 - G_2\Delta_1,\ \phi_2 = (G_1)^{-1}\Gamma_2^6(\Gamma_2^4)^{-1} - G_2\Delta_2,\ \phi_3 = \frac{K_1G_2}{\delta} - (G_1)^{-1},$$

$$\Delta_1 = \frac{1}{\delta}(K_1G_3 - G_1K_3),\ \Delta_2 = \frac{1}{\delta}(K_1\Gamma_2^6 - G_1\Gamma_2^5)(\Gamma_2^4)^{-1},\ \delta = K_1G_2 - K_2G_1,$$

$$K_1 = \Gamma_3^5 - \Gamma_2^5(\Gamma_2^4)^{-1}\Gamma_3^4,\ K_2 = \Gamma_4^5 - \Gamma_2^5(\Gamma_2^4)^{-1}\Gamma_4^4,\ K_3 = Z_2 - \Gamma_2^5(\Gamma_2^4)^{-1}Z_3$$

$$G_1 = \Gamma_3^6 - \Gamma_2^6(\Gamma_2^4)^{-1}\Gamma_3^4,\ G_2 = \Gamma_4^6 - \Gamma_2^6(\Gamma_2^4)^{-1}\Gamma_4^4,\ G_3 = Z_1 - \Gamma_2^6(\Gamma_2^4)^{-1}Z_3,$$

$$Z_1 = \lambda_0 - \epsilon_2\Gamma_5^6 - \epsilon_1\Gamma_6^6 - \epsilon_0,\ Z_2 = \lambda_1 - \epsilon_2\Gamma_5^5 - \epsilon_1,\ Z_3 = \lambda_2 - \epsilon_2,$$

$$\Gamma_k^j = \frac{1}{\Gamma(\sum_{i=k}^{j}\alpha_i + 1)}.$$

Proof: We prove the first implication.
We utilise Lemma 2.4, we observe that

$$D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t) = I^{\alpha_1}S(t) + l_0,$$

$$D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t) = I^{\alpha_1+\alpha_2}S(t) + l_0\frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + l_1,$$

$$D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t) = I^{\alpha_1+\alpha_2+\alpha_3}S(t) + l_0\frac{t^{\alpha_2+\alpha_3}}{\Gamma(\alpha_2+\alpha_3+1)} + l_1\frac{t^{\alpha_3}}{\Gamma(\alpha_3+1)} + l_2,$$

$$\begin{aligned} D^{\alpha_5}D^{\alpha_6}x(t) &= I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4}S(t) + l_0\frac{t^{\alpha_2+\alpha_3+\alpha 4}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+1)} + l_1\frac{t^{\alpha_3+\alpha_4}}{\Gamma(\alpha_3+\alpha_4+1)} \\ &+ l_2\frac{t^{\alpha_4}}{\Gamma(\alpha_4+1)} + l_3, \end{aligned}$$

$$\begin{aligned} D^{\alpha_6}x(t) &= I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}S(t) + l_0\frac{t^{\alpha_2+\alpha_3+\alpha 4+\alpha_5}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+\alpha_5+1)} + l_1\frac{t^{\alpha_3+\alpha_4+\alpha_5}}{\Gamma(\alpha_3+\alpha_4+\alpha_5+1)} \\ &+ l_2\frac{t^{\alpha_4+\alpha_5}}{\Gamma(\alpha_4+\alpha_5+1)} + l_3\frac{t^{\alpha_5}}{\Gamma(\alpha_5+1)} + l_4, \end{aligned}$$

$$\begin{aligned} x(t) &= I^{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}S(t) + l_0\frac{t^{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}}{\Gamma(\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+1)} \\ &+ l_1\frac{t^{\alpha_3+\alpha_4+\alpha_5+\alpha_6}}{\Gamma(\alpha_3+\alpha_4+\alpha_5+\alpha_6+1)} + l_2\frac{t^{\alpha_4+\alpha_5+\alpha_6}}{\Gamma(\alpha_4+\alpha_5+\alpha_6+1)} + l_3\frac{t^{\alpha_5+\alpha_6}}{\Gamma(\alpha_5+\alpha_6+1)} \\ &+ l_4\frac{t^{\alpha_6}}{\Gamma(\alpha_6+1)} + l_5, \end{aligned}$$

we have

$$x(0) = \epsilon_0 \Rightarrow \ l_5 = \epsilon_0$$

$$D^{\alpha_6}\ x(0) = \epsilon_1 \Rightarrow \ l_4 = \epsilon_1$$

$$D^{\alpha_5}D^{\alpha_6}\ x(0) = \epsilon_2 \Rightarrow \ l_3 = \epsilon_2$$

By considering

$$\begin{aligned} &x(1) = \lambda_0, \\ &D^{\alpha_6}\ x(1) = \lambda_1, \\ &D^{\alpha_5}D^{\alpha_6}\ x(1) = \lambda_2, \end{aligned}$$

we get:

$$l_2 = \left[\Delta_1 + \Delta_2 I^{\sum_{i=1}^{4}\alpha_i} S(1) + \frac{G_1}{\delta} I^{\sum_{i=1}^{5}\alpha_i} S(1) - \frac{K_1}{\delta} I^{\sum_{i=1}^{6}\alpha_i} S(1) \right],$$

$$l_1 = \left[\phi_1 + \phi_2 I^{\sum_{i=1}^{4}\alpha_i} S(1) + \phi_3 I^{\sum_{i=1}^{6}\alpha_i} S(1) - \frac{G_1 G_2}{\delta} I^{\sum_{i=1}^{5}\alpha_i} S(1) \right],$$

$$l_0 = ((\Gamma_2^4)^{-1}) \left[\varphi_1 - \varphi_2 I^{\sum_{i=1}^{4}\alpha_i} S(1) + \varphi_3 I^{\sum_{i=1}^{5}\alpha_i} S(1) + \varphi_4 I^{\sum_{i=1}^{6}\alpha_i} S(1) \right].$$

We achieve the proof.
In what follows, we need both

$$B := \{x \in C(J,\mathbb{R}), D^{\alpha_5} D^{\alpha_6} x \in C(J,\mathbb{R}), D^{\alpha_3} D^{\alpha_4} D^{\alpha_5} D^{\alpha_6} x \in C(J,\mathbb{R})\},$$

and

$$\|x\|_B = \|x\|_\infty + \|D^{\alpha_5} D^{\alpha_6} x\|_\infty + \|D^{\alpha_3} D^{\alpha_4} D^{\alpha_5} D^{\alpha_6} x\|_\infty$$

where,

$$\|x\|_\infty = \sup_{t\in J} |x(t)| \ , \|D^{\alpha_5} D^{\alpha_6} x\|_\infty = \sup_{t\in J} |D^{\alpha_5} D^{\alpha_6} x(t)|.$$
$$\|D^{\alpha_3} D^{\alpha_4} D^{\alpha_5} D^{\alpha_6} x\|_\infty = \sup_{t\in J} |D^{\alpha_3} D^{\alpha_4} D^{\alpha_5} D^{\alpha_6} x(t)|.$$

Then, we consider the application $U : B \to B$, such that

$$\begin{aligned} Ux(t) \ &= \ I^{\sum_{i=1}^{6}\alpha_i} S_x^*(t) + (\Gamma_2^4)^{-1}\Gamma_2^6 \Bigg[\varphi_1 - \varphi_2 I^{\sum_{i=1}^{4}\alpha_i} S_x^*(1) + \varphi_3 I^{\sum_{i=1}^{5}\alpha_i} S_x^*(1) \\ &+\varphi_4 I^{\sum_{i=1}^{6}\alpha_i} S_x^*(1) \Bigg] t^{\sum_{i=2}^{6}\alpha_i} + \Gamma_3^6 \Bigg[\phi_1 + \phi_2 I^{\sum_{i=1}^{4}\alpha_i} S_x^*(1) + \phi_3 I^{\sum_{i=1}^{6}\alpha_i} S_x^*(1) \\ &-\frac{G_1 G_2}{\delta} I^{\sum_{i=1}^{5}\alpha_i} S_x^*(1) \Bigg] t^{\sum_{i=3}^{6}\alpha_i} + \Gamma_4^6 \Bigg[\Delta_1 + \Delta_2 I^{\sum_{i=1}^{4}\alpha_i} S_x^*(1) + \frac{G_1}{\delta} I^{\sum_{i=1}^{5}\alpha_i} S(1) \\ &-\frac{K_1}{\delta} I^{\sum_{i=1}^{6}\alpha_i} S_x^*(1) \Bigg] t^{\sum_{i=4}^{6}\alpha_i} + \epsilon_2 \Gamma_5^6 t^{\sum_{i=5}^{6}\alpha_i} + \epsilon_1 \Gamma_6^6 t^{\alpha_6} + \epsilon_0 \end{aligned}$$

where

$$S_x^*(t) = m_1(t, x(t), D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t)) + m_2(t, x(t), D^{\alpha_5}D^{\alpha_6}x(t)) + m_3(t, x(t), I^{\xi}x(t)) + m_4(t, x(t)).$$

3 Main results

We shall consider what follows:

$(\varpi 1)$: We suppose that m_1, m_2 and m_3 are defined on $[0,1]\times\mathbb{R}^2$ and continuous, and m_4 is defined on $[0,1]\times\mathbb{R}$ and continuous.

$(\varpi 2)$: There exist some functions $n_i, z_i, \theta_i, i = 1, 2,$ such that for any t $\in J$, $x_i, y_i \in \mathbb{R}, i = 1, 2,$

$$\begin{aligned}
|m_1(t,\ x_1,\ x_2) - m_1(t,\ y_1,\ y_2)| &\leq \sum_{i=1}^{2} n_i(t)|\ x_i -\ y_i|,\\
|m_2(t,\ x_1,\ x_2) - m_2(t,\ y_1,\ y_2)| &\leq \sum_{i=1}^{2} z_i(t)|\ x_i -\ y_i|,\\
|m_3(t,\ x_1,\ x_2) - m_3(t,\ y_1,\ y_2)| &\leq \sum_{i=1}^{2} \theta_i(t)|\ x_i -\ y_i|,
\end{aligned}$$

$(\varpi 3)$: There exist a continuous function p, for any $t \in J$, $x, y \in \mathbb{R}$,

$$|m_4(t,\ x) - m_4(t,\ y)| \leq p(t)|\ x -\ y|.$$

We suppose:

$$n^* = \max\{\sup_{t\in J}|n_1(t)|, \sup_{t\in J}|n_2(t)|\} \qquad z^* = \max\{\sup_{t\in J}|z_1(t)|, \sup_{t\in J}|z_2(t)|\}$$
$$\theta^* = \max\{\sup_{t\in J}|\theta_1(t)|, \sup_{t\in J}|\theta_2(t)|\} \qquad p^* = \sup_{t\in J}|p(t)|.$$

3.1 Banach Contraction Principle for a Unique Solution

First, let us put

$$\begin{aligned}
\mu_1 &= \Gamma_1^6\Upsilon + \Upsilon(\Gamma_2^4)^{-1}\Gamma_2^6\left(|\varphi_2|\Gamma_1^4 + |\varphi_3|\Gamma_1^5 + |\varphi_4|\Gamma_1^6\right) + \Gamma_3^6\Upsilon\left(|\phi_2|\Gamma_1^4 + |\phi_3|\Gamma_1^6 + |\frac{G_1G_2}{\delta}|\Gamma_1^5\right)\\
&\quad +\Gamma_4^6\Upsilon\left(|\Delta_2|\Gamma_1^4 + |\frac{G_1}{\delta}|\Gamma_1^5 + |\frac{K_1}{\delta}|\Gamma_1^6\right)
\end{aligned}$$

$$\mu_2 = \Gamma_1^4\Upsilon + \Upsilon(\Gamma_2^4)^{-1}\Gamma_2^4\Bigg(|\varphi_2|\Gamma_1^4 + |\varphi_3|\Gamma_1^5 + |\varphi_4|\Gamma_1^6\Bigg) + \Gamma_3^4\Upsilon\Bigg(|\phi_2|\Gamma_1^4 + |\phi_3|\Gamma_1^6 + |\frac{G_1G_2}{\delta}|\Gamma_1^5\Bigg)$$
$$+\Gamma_4^4\Upsilon\Bigg(|\Delta_2|\Gamma_1^4 + |\frac{G_1}{\delta}|\Gamma_1^5 + |\frac{K_1}{\delta}|\Gamma_1^6\Bigg)$$

$$\mu_3 = \Gamma_1^2\Upsilon + \Upsilon(\Gamma_2^4)^{-1}\Gamma_2^2\Bigg(|\varphi_2|\Gamma_1^4 + |\varphi_3|\Gamma_1^5 + |\varphi_4|\Gamma_1^6\Bigg) + \Upsilon\Bigg(|\Delta_2|\Gamma_1^4 + |\frac{G_1}{\delta}|\Gamma_1^5 + |\frac{K_1}{\delta}|\Gamma_1^6\Bigg)$$

$$\Upsilon = \Bigg(\theta^* + n^* + z^* + \tfrac{\theta^*}{\Gamma(\zeta+1)} + p\Bigg)$$

where $\delta \neq 0$.
Now, we pass to establish the following result:

Theorem 3.1 *Assume that* $(\varpi_1), (\varpi_2), (\varpi_3)$ *are satisfied. Then,* (1) *has a unique solution if* $\sum_{i=1}^{3}\mu_i \in]0,1[$.

Proof:
Let $(x,y) \in B^2$, we can write

$$\begin{aligned}
&\|U(x) - U(y)\|_\infty \\
\leq\ & \Gamma_1^6\Bigg(\theta^* + n^* + z^* + \tfrac{\theta^*}{\Gamma(\zeta+1)} + p\Bigg)\|x-y\|_B \\
&+\Bigg(\theta^* + n^* + z^* + \tfrac{\theta^*}{\Gamma(\zeta+1)} + p\Bigg)(\Gamma_2^4)^{-1}\Gamma_2^6\Bigg(|\varphi_2|\Gamma_1^4 + |\varphi_3|\Gamma_1^5 + |\varphi_4|\Gamma_1^6\Bigg)\|x-y\|_B \\
&+\Bigg(\theta^* + n^* + z^* + \tfrac{\theta^*}{\Gamma(\zeta+1)} + p\Bigg)\Gamma_3^6\Bigg(|\phi_2|\Gamma_1^4 + |\phi_3|\Gamma_1^6 + |\frac{G_1G_2}{\delta}|\Gamma_1^5\Bigg)\|x-y\|_B \\
&+\Gamma_4^6\Bigg(\theta^* + n^* + z^* + \tfrac{\theta^*}{\Gamma(\zeta+1)} + p\Bigg)\Bigg(|\Delta_2|\Gamma_1^4 + |\frac{G_1}{\delta}|\Gamma_1^5 + |\frac{K_1}{\delta}|\Gamma_1^6\Bigg)\|x-y\|_B \\
\leq\ & \Bigg(\theta^* + n^* + z^* + \tfrac{\theta^*}{\Gamma(\zeta+1)} + p\Bigg)\Bigg[\Gamma_1^6 + (\Gamma_2^4)^{-1}\Gamma_2^6\Bigg(|\varphi_2|\Gamma_1^4 + |\varphi_3|\Gamma_1^5 + |\varphi_4|\Gamma_1^6\Bigg) \\
&+\Gamma_3^6\Bigg(|\phi_2|\Gamma_1^4 + |\phi_3|\Gamma_1^6 + |\frac{G_1G_2}{\delta}|\Gamma_1^5\Bigg) + \Gamma_4^6\Bigg(|\Delta_2|\Gamma_1^4 + |\frac{G_1}{\delta}|\Gamma_1^5 + |\frac{K_1}{\delta}|\Gamma_1^6\Bigg)\Bigg]\|x-y\|_B \\
\leq\ & \mu_1\|x-y\|_B
\end{aligned}$$

.

$$\begin{aligned}
D^{\alpha_5}D^{\alpha_6}Ux(t) &= I^{\sum_{i=1}^{4}\alpha_i}S_x^*(t)+(\Gamma_2^4)^{-1}\Gamma_2^4\Big[\varphi_1-\varphi_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)+\varphi_3 I^{\sum_{i=1}^{5}\alpha_i}S_x^*(1)\\
&+\varphi_4 I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\Big]t^{\sum_{i=2}^{4}\alpha_i}+\Gamma_3^4\Big[\phi_1+\phi_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)+\phi_3 I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\\
&-\frac{G_1G_2}{\delta}I^{\sum_{i=1}^{5}\alpha_i}S_x^*(1)\Big]t^{\sum_{i=3}^{4}\alpha_i}+\Gamma_4^4\Big[\Delta_1+\Delta_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)\\
&+\frac{G_1}{\delta}I^{\sum_{i=1}^{5}\alpha_i}S(1)-\frac{K_1}{\delta}I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\Big]t^{\alpha_4}+\epsilon_2
\end{aligned}$$

$$\begin{aligned}
\|D^{\alpha_5}D^{\alpha_6}U(x)-D^{\alpha_5}D^{\alpha_6}U(y)\|_\infty &\leq \Gamma_1^4\Upsilon\|x-y\|_B\\
&+\Upsilon(\Gamma_2^4)^{-1}\Gamma_2^4\Big(|\varphi_2|\Gamma_1^4+|\varphi_3|\Gamma_1^5+|\varphi_4|\Gamma_1^6\Big)\|x-y\|_B\\
&+\Upsilon\Gamma_3^4\Big(|\phi_2|\Gamma_1^4+|\phi_3|\Gamma_1^6+|\frac{G_1G_2}{\delta}|\Gamma_1^5\Big)\|x-y\|_B\\
&+\Gamma_4^4\Upsilon\Big(|\Delta_2|\Gamma_1^4+|\frac{G_1}{\delta}|\Gamma_1^5+|\frac{K_1}{\delta}|\Gamma_1^6\Big)\|x-y\|_B\\
&\leq \Upsilon\Big[\Gamma_1^4+(\Gamma_2^4)^{-1}\Gamma_2^4\Big(|\varphi_2|\Gamma_1^4+|\varphi_3|\Gamma_1^5+|\varphi_4|\Gamma_1^6\Big)\\
&+\Gamma_3^4\Big(|\phi_2|\Gamma_1^4+|\phi_3|\Gamma_1^6+|\frac{G_1G_2}{\delta}|\Gamma_1^5\Big)+\Gamma_4^4\Big(|\Delta_2|\Gamma_1^4\\
&+|\frac{G_1}{\delta}|\Gamma_1^5+|\frac{K_1}{\delta}|\Gamma_1^6\Big)\Big]\|x-y\|_B\\
&\leq \mu_2\|x-y\|_B
\end{aligned}$$

.

$$
\begin{aligned}
D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}Ux(t) &= I^{\sum_{i=1}^{2}\alpha_i}S_x^*(t)+(\Gamma_2^4)^{-1}\Gamma_2^2\Bigg[\varphi_1-\varphi_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)\\
&+\varphi_3 I^{\sum_{i=1}^{5}\alpha_i}S_x^*(1)+\varphi_4 I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\Bigg]t^{\alpha_2}+\Gamma_3^4\Bigg[\phi_1\\
&+\phi_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)+\phi_3 I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)-\frac{G_1G_2}{\delta}I^{\sum_{i=1}^{5}\alpha_i}S_x^*(1)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\|D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}U(x)-D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}U(y)\|_\infty\\
\leq\ & \Gamma_1^2\Upsilon\|x-y\|_B+\Upsilon(\Gamma_2^4)^{-1}\Gamma_2^2\Bigg(|\varphi_2|\Gamma_1^4+|\varphi_3|\Gamma_1^5+|\varphi_4|\Gamma_1^6\Bigg)\|x-y\|_B\\
&+\Upsilon\Bigg(|\phi_2|\Gamma_1^4+|\phi_3|\Gamma_1^6+|\frac{G_1G_2}{\delta}|\Gamma_1^5\Bigg)\|x-y\|_B\\
\leq\ & \Upsilon\Bigg[\Gamma_1^2+(\Gamma_2^4)^{-1}\Gamma_2^2\Bigg(|\varphi_2|\Gamma_1^4+|\varphi_3|\Gamma_1^5+|\varphi_4|\Gamma_1^6\Bigg)\\
&+\Bigg(|\phi_2|\Gamma_1^4+|\phi_3|\Gamma_1^6+|\frac{G_1G_2}{\delta}|\Gamma_1^5\Bigg)\Bigg]\|x-y\|_B\\
\leq\ & \mu_3\|x-y\|_B
\end{aligned}
$$

.

Consequently, we observe that

$$\|U(x)-U(y)\|_B \leq (\mu_1+\mu_2+\mu_3)\|x-y\|_B.$$

Hence, by Banach fixed point theorem, F has a unique fixed point which is the unique solution of (1).

3.2 An Ulam Hyers Stability Result

First, we introduce the following definition related to our problem.

Definition 3.2 *The equation* (1) *has the Ulam Hyers stability if there exists a real number* $\rho > 0$, *such that for each* $\varrho > 0, t \in [0,1]$ *and for each* $x \in B$

solution of the inequality

$$\begin{aligned}&\big|D^{\alpha_1}D^{\alpha_2}D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t)-m_1(t,x(t),D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}x(t))\\&-m_2(t,x(t),D^{\alpha_5}D^{\alpha_6}x(t))-m_3(t,x(t),I^{\xi}x(t))-m_4(t,x(t))|\leq\varrho,\end{aligned}\tag{3}$$

under the following conditions:

$$\begin{cases}x(0)=\epsilon_0,\quad x(1)=\lambda_0,\quad \epsilon_0,\ \lambda_0\in\mathbb{R},\\ D^{\alpha_6}x(0)=\epsilon_1,\quad D^{\alpha_6}x(1)=\lambda_1,\quad \epsilon_1,\ \lambda_1\in\mathbb{R},\\ D^{\alpha_5}D^{\alpha_6}x(0)=\epsilon_2,\quad D^{\alpha_5}D^{\alpha_6}x(1)=\lambda_2,\quad \epsilon_2,\ \lambda_2\in\mathbb{R},\end{cases}$$

there exists $x^*\in B$ *a solution of* (1), *such that*

$$\|x-x^*\|_B\leq\rho\varrho.$$

Definition 3.3 *The equation* (1) *has the Ulam Hyers stability in the generalized sense if there exists* $\rho\in C(\mathbb{R}^+,\mathbb{R}^+)$*;* $\rho(0)=0$*, such that for each* $\varrho>0$*, and for any* $x\in B$ *solution of* (3)*, there exists a solution* $x^*\in B$ *of* (1)*, such that*

$$\|x-x^*\|_B<\rho(\varrho).$$

Now, we propose the following theorem

Theorem 3.4 *The conditions of Theorem* (3.1) *allow us to state that problem* (1) *is Ulam Hyers stable.*

Proof: Let $x\in B$ be a solution of (3), and let, by Theorem 3.1, $x^*\in B$ be the unique solution of (1).
By integration of (3), we obtain

$$\begin{aligned}&\Bigg|x(t)-I^{\sum_{i=1}^{6}\alpha_i}S_x^*(t)-(\Gamma_2^4)^{-1}\Gamma_2^6\Bigg[\varphi_1-\varphi_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)+\varphi_3 I^{\sum_{i=1}^{5}\alpha_i}S_x^*(1)+\\&\varphi_4 I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\Bigg]t^{\sum_{i=2}^{6}\alpha_i}-\Gamma_3^6\Bigg[\phi_1+\phi_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)+\phi_3 I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\\&-\frac{G_1G_2}{\delta}I^{\sum_{i=1}^{5}\alpha_i}S_x^*(1)\Bigg]t^{\sum_{i=3}^{6}\alpha_i}-\Gamma_4^6\Bigg[\Delta_1+\Delta_2 I^{\sum_{i=1}^{4}\alpha_i}S_x^*(1)+\frac{G_1}{\delta}I^{\sum_{i=1}^{5}\alpha_i}S(1)\\&-\frac{K_1}{\delta}I^{\sum_{i=1}^{6}\alpha_i}S_x^*(1)\Bigg]t^{\sum_{i=4}^{6}\alpha_i}-\epsilon_2\Gamma_5^6 t^{\sum_{i=5}^{6}\alpha_i}-\epsilon_1\Gamma_6^6 t^{\alpha_6}-\epsilon_0\Bigg|\leq\varrho\Gamma_1^6\end{aligned}\tag{4}$$

.

Using (3) and (4), we get

$$\|x - x^*\|_\infty \leq \varrho\Gamma_1^6 + \mu_1\|x - x^*\|_\infty.$$

Also, we have

$$\|D^{\alpha_5}D^{\alpha_6}(x - x^*)\|_\infty \leq \varrho\Gamma_1^4 + \mu_2\|D^{\alpha_5}D^{\alpha_6}(x - x^*)\|_\infty.$$

and

$$\|D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}(x - x^*)\|_\infty \leq \varrho\Gamma_1^2 + \mu_3\|D^{\alpha_3}D^{\alpha_4}D^{\alpha_5}D^{\alpha_6}(x - x^*)\|_\infty.$$

So

$$\|x - x^*\|_B \leq \varrho(\Gamma_1^6 + \Gamma_1^4 + \Gamma_1^2) + (\mu_1 + \mu_2 + \mu_3)\|x - x^*\|_B,$$

$$\|x - x^*\|_B \leq \frac{\varrho(\Gamma_1^6 + \Gamma_1^4 + \Gamma_1^2)}{1 - \sum_{i=1}^{3}\mu_i}.$$

Thus,

$$\|x - x^*\|_B \leq \rho\varrho,$$

where

$$\rho = \frac{\Gamma_1^6 + \Gamma_1^4 + \Gamma_1^2}{1 - \sum_{i=1}^{3}\mu_i}.$$

Thus, (1) has the Ulam Hyers stability.

Remark 3.5 *When $\rho(\varrho) = \varrho.\rho$, we have the generalised Ulam Hyers stability for* (1)*.*

Example 3.6 *Consider the problem:*

$$\begin{cases} \mathcal{D}^{\frac{1}{2}}\mathcal{D}^{\frac{1}{3}}\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{2}{5}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t) = \dfrac{(1+x(t))}{e\sqrt{t}}\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{2}{5}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t) + \dfrac{(1-x(t))\cos t}{1+\sqrt{t}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t) \\ +\dfrac{x(t)}{1+\ln t}\mathcal{I}^{\frac{1}{2}}x(t) + \sin t + x(t), \qquad t \in [0,1], \\ x(0) = 1, \quad x(1) = e, \\ \mathcal{D}^{\frac{4}{5}}x(0) = \sqrt{2}, \quad \mathcal{D}^{\frac{4}{5}}x(1) = 1 \\ \mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(0) = \sqrt{3}, \qquad \mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(0) = \dfrac{1}{2}. \end{cases} \tag{5}$$

We have

$$\begin{cases} m_1(t,x(t),\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{2}{5}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t)) = \dfrac{(1+x(t))}{e\sqrt{t}}\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{2}{5}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t), \\ m_2(t,x(t),\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t)) = \dfrac{(1-x(t))\cos t}{1+\sqrt{t}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}x(t), \\ m_3(t,x(t),\mathcal{I}^{\frac{1}{2}}x(t)) = \dfrac{x(t)}{1+\ln t}\mathcal{I}^{\frac{1}{2}}x(t), \\ m_4(t,x(t)) = \sin t + x(t). \end{cases}$$

These functions are continuous over $[0,1]$. *Also, one can see that*

$$\begin{cases} \alpha_1 = \dfrac{1}{2},\ \alpha_2 = \dfrac{1}{3},\ \alpha_3 = \dfrac{1}{4},\ \alpha_4 = \dfrac{2}{3},\ \alpha_5 = \dfrac{3}{4},\ \alpha_6 = \dfrac{4}{5}, \\ \epsilon_0 = 1,\ \epsilon_1 = \sqrt{2},\ \epsilon_2 = \sqrt{3},\ \lambda_0 = e,\ \lambda_1 = 1,\ \lambda_2 = \dfrac{1}{2},\ \xi = \dfrac{1}{2}. \end{cases}$$

Also, for all $t \in [0,1]$, *we can write*

$$|m_1(.,x_1,x_2) - m_1(.,y_1,y_2)| \leq \frac{|x_2|}{e\sqrt{t}}|x_1-y_1| + \frac{|1+y_1|}{e\sqrt{t}}|x_2-y_2|,$$

then

$$n_1(t) = \frac{|x_2|}{e\sqrt{t}}, \quad n_2(t) = \frac{|1+y_1|}{e\sqrt{t}}$$

and

$$\begin{cases} |m_2(.,x_1,x_2) - m_2(.,y_1,y_2)| \leq \dfrac{|\cos t|}{1+\sqrt{t}}|x_1-y_1| + \dfrac{|\cos t|}{1+\sqrt{t}}|x_2-y_2|, \\ |y_2| < 1. \end{cases}$$

Therefore,

$$z_1(t) = z_2(t) = \frac{|\cos t|}{1+\sqrt{t}}.$$

Using the same arguments as before, we get

$$\begin{cases} |m_3(t,x_1,x_2) - m_3(t,y_1,y_2)| \leq \dfrac{1}{1+\ln t}|x_1-y_1| + \dfrac{1}{1+\ln t}|x_2-y_2|, \\ |x_1| < 1,\ \ |y_2| < 1. \end{cases}$$

Hence,

$$\theta_1(t) = \theta_2(t) = \frac{1}{1+\ln t}.$$

In addition, we have

$$|m_4(.,x_1) - m_4(.,y_1)| \leq |x_1-y_1|,$$

where

$$p(t) = 1.$$

On other hand, we have

$$\mu_1 = 0,101, \quad \mu_2 = 0,019, \quad \mu_3 = 0,007,$$

then

$$\sum_{k}^{3} \mu_k = 0,127 \in \left]0,1\right[.$$

Hence, (5) *has a unique solution on* $[0,1]$.

Example 3.7 *Consider the following second example:*

$$\begin{cases} \mathcal{D}^{\frac{11}{12}}\mathcal{D}^{\frac{11}{13}}\mathcal{D}^{\frac{11}{14}}\mathcal{D}^{\frac{12}{15}}\mathcal{D}^{\frac{13}{14}}\mathcal{D}^{\frac{14}{15}}y(t) = \dfrac{y(t)}{\pi + \sin 7}\left(\mathcal{D}^{\frac{1}{4}}\mathcal{D}^{\frac{2}{5}}\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}\right)y(t) \\ +\left(\dfrac{y(t)}{1+\sqrt{2}} + 100\right)\left(\mathcal{D}^{\frac{3}{4}}\mathcal{D}^{\frac{4}{5}}\right)y(t) \\ +y(t)\mathcal{I}^{1}y(t) + \dfrac{y(t)}{\ln t + e}, \qquad t \in [0,1], \\ x(0) = \dfrac{1}{\pi}, \quad x(1) = \sqrt{e}, \\ \mathcal{D}^{\frac{14}{15}}y(0) = \dfrac{\sqrt{2}}{2}, \quad \mathcal{D}^{\frac{14}{15}}y(1) = \dfrac{1}{1+\ln 5} \\ \mathcal{D}^{\frac{13}{14}}\mathcal{D}^{\frac{14}{15}}y(0) = \dfrac{1}{\pi+\sqrt{3}}, \qquad \mathcal{D}^{\frac{13}{14}}\mathcal{D}^{\frac{14}{15}}y(0) = \dfrac{\sqrt{2}}{100+\sqrt{3}}, \end{cases} \tag{6}$$

where

$$\alpha_1 = \frac{11}{12},\ \alpha_2 = \frac{11}{13},\ \alpha_3 = \frac{11}{14},\ \alpha_4 = \frac{12}{13},\ \alpha_5 = \frac{13}{14},\ \alpha_6 = \frac{14}{15},$$

and

$$\epsilon_0 = \frac{1}{\pi},\ \epsilon_1 = \frac{\sqrt{2}}{2},\ \epsilon_2 = \frac{1}{\pi+\sqrt{3}},\ \lambda_0 = \sqrt{e},\ \lambda_1 = \frac{1}{1+\ln 5},\ \lambda_2 = \frac{\sqrt{2}}{100+\sqrt{3}},\ \xi = 1.$$

We have

$$\sum_{i}^{3} \mu_i = 0,0071859 < 1.$$

Consequently, (6) *has a unique solution on* $[0,1]$.

4 Open Problem

1: Is it possible to examine the Ulam-Hyers stability for the above (regular) class of nonlinear fractional differential equations by introducing a singular perturbed term?
2: What can happened when we compare the regular (initial) problem solutions with those of the perturbed one?

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