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# Submanifold of a K-contact manifold admitting Zamkovoy connection

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**Abstract**

***The main objective of this paper is to investigate $\eta$-Ricci solitons on anti-invariant submanifolds of a K-contact manifold, specifically with respect to Zamkovoy connection. We examine the characteristics of anti-invariant submanifold of a K-contact metric admitting Zamkovoy connection. Further, we analyse the behavior of a $\eta$-Ricci soliton on anti-invariant submanifold.***

**Keywords:** *Anti-invariant submanifold, $\eta$-Ricci soliton, Zamkovoy connection, Concircular curvature tensor, $M$-projective curvature tensor.*

**2010 Mathematics Subject Classification:** 53C15, 53C25, 53C40, 53D15.

## 1 Introduction

In recent years, geometric flow, especially the Ricci flow, have gained significant interest as a research topic in differential geometry. low is referred to as a Ricci soliton if its evolution is determined solely by a one-parameter group of diffeomorphisms and scaling. Hamilton introduced the concept of Ricci solitons [9]. A Ricci soliton $(g, V, \lambda)$ on a Riemannian manifold $(\tilde{M}, g)$ is a generalization of Einstein metric given by [9]

$$L_V g + 2S + 2\lambda g = 0,$$

where $L_V$ represents the Lie derivative along the vector field $V$ on $\tilde{M}$, $S$ denotes the Ricci tensor and $\lambda$ is a real number. The Ricci soliton is classified as shrinking, steady, or expanding depending on whether $\lambda$ is negative, zero, or positive, respectively.
In [8], Cho and Kimura, introduced the concept of $\eta$-Ricci solitons for real hypersurfaces in a non-flat complex space form. The Riemannian metric $g$ is classified as a $\eta$-Ricci soliton if there exists a potential vector field $V$ and constants $\lambda$, $\mu$, satisfying the equation

$$L_V g + 2S(X,Y) + 2\lambda g(X,Y) + 2\eta \otimes \eta = 0. \tag{1}$$

$\eta$-Ricci solitons have been extensively studied by Blaga [6], Prakasha et al. [16] and other researchers.

In 2009, Zamkovoy [20] introduced the notion of Zamkovoy connection. Subsequently, this concept has been studied by Biswas and Baishya ([1], [2]), Blaga [6], Mandal and Das ([11], [12]) and many other researchers.

Yano introduced the notion of concircular curvature tensor $C$ of type (1, 3) on Riemannian manifold for an $n$-dimensional manifold $\tilde{M}$ as given in [18]

$$C(X,Y)Z = \tilde{R}(X,Y)Z - \frac{r}{n(n-1)}\left[g(Y,Z)X - g(X,Z)Y\right],$$

for all vector fields $X, Y, Z \in \chi(\tilde{M})$, where $R$ is the Riemannian curvature tensor of type (1, 3) and $r$ is the scalar curvature.
Therefore, considering $C^*$ as the concircular curvature tensor with respect to Zamkovoy connection, for a $(2n+1)$-dimensional manifold $\tilde{M}$, we express it as:

$$C^*(X,Y)Z = R^*(X,Y)Z - \frac{r^*}{2n(2n+1)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{2}$$

for all vector fields $X, Y, Z, \in \chi(\tilde{M})$, where $R^*$ is the Riemannian curvature tensor and $r^*$ is the scalar curvature with respect to Zamkovoy connection.

**Definition 1.1** *[10] A $(2n+1)$-dimensional manifold $\tilde{M}$ is termed Ricci flat if $S(X,Y) = 0$, for all $X, Y, Z \in \chi(\tilde{M})$.*

**Definition 1.2** *[12] A $(2n+1)$-dimensional manifold $\tilde{M}$ is termed concircularly flat with respect to Zamkovoy connection if $C^*(X,Y)Z = 0$, for all $X, Y, Z \in \chi(\tilde{M})$.*

In 1971, Pokhariyal and Mishra [15] introduced the concept of a M-projective curvature tensor on a Riemannian manifold of dimension $n$. This tensor, denoted by $M$, is defined as follows:

$$\begin{aligned} M(X,Y)Z =& \tilde{R}(X,Y)Z - \frac{1}{2(n-1)}\left[S(Y,Z)X - S(X,Z)Y\right] \\ &- \frac{1}{2(n-1)}\left[g(Y,Z)QX - g(X,Z)QY\right], \end{aligned}$$

for all $X, Y, Z, \in \chi(\tilde{M})$, where $Q$ is the Ricci operator.
Considering $M^*$ as the M-projective curvature tensor with respect to Zamkovoy connection and for a $(2n+1)$-dimensional manifold, we get

$$\begin{aligned} M^*(X,Y)Z =& R^*(X,Y)Z - \frac{1}{4n}\left[S^*(Y,Z)X - S^*(X,Z)Y\right] \\ & - \frac{1}{4n}\left[g(Y,Z)Q^*X - g(X,Z)Q^*Y\right], \end{aligned} \tag{3}$$

for all $X, Y, Z, \in \chi(\tilde{M})$, where $Q^*$ is the Ricci operator with respect to Zamkovoy connection.

**Definition 1.3** *[11] A $(2n+1)$-dimensional manifold $\tilde{M}$ is termed M-projectively flat admitting Zamkovoy connection if $M^*(X,Y)Z = 0$, for all $X, Y, Z \in \chi(\tilde{M})$.*

Let $\psi$ be a differentiable map from a manifold $N$ of dimension $n$ into a manifold $\tilde{M}$ of dimension $m$, respectively. If at each point $u$ of $N$, the differential $(\psi_*)_u$ is injective (i.e., rank$\psi$=$n$), then $\psi$ is termed an immersion of $N$ into $\tilde{M}$.

If an immersion $\psi$ is injective, meaning $\psi(u) \neq \psi(v)$ for $u \neq v$, then $\psi$ is reffered to as an imbedding of $N$ into $\tilde{M}$.
If the manifolds $N$, $\tilde{M}$ satisfy the following two conditions, then $N$ is called a submanifold of $\tilde{M}$ [10]

- $N$ is a subset of $\tilde{M}$,
- The inclusion map $i$ from $N$ into $\tilde{M}$ is an imbedding.

A submanifold $N$ is said to be anti-invariant if $X \in T_xN$ and $\phi X \in T_x^{\perp}N$, for all $x \in N$, where $T_xN$ and $T_x^{\perp}N$ represent the tangent space and the normal space of $N$. Hence, in an anti-invariant submanifold $N$, we have [19]

$$g(X, \phi Y) = 0, \tag{4}$$

for all $X, Y \in \chi(TN)$.

In 1977, Yano and Kon examined anti-invariant submanifolds of Sasakian manifolds [19]. Meanwhile, Pandey and Kumar [14] investigated on anti-invariant submanifolds of almost para-contact manifolds. Additionally, Shivaprasanna et al. [17] explored Ricci-Yamabe solitons on anti-invariant submanifolds of certain indefinite almost contact manifolds. Recently, Karmakar [10] studied $\eta$-Ricci-Yamabe solitons on anti-invariant submanifolds of trans Sasakian manifold equipped with the Zamkovoy connection.

Motivated by the aforementioned studies, this paper investigates $\eta$-Ricci solitons on anti-invariant submanifolds of a K-contact manifold admitting Zamkovoy connection. Additionally, we examine $\eta$-Ricci solitons on these submanifolds under conditions of Ricci flatness, concircular flatness, and M-projective flatness with respect to Zamkovoy connection. Our findings reveal the nature of the $\eta$-Ricci soliton, whether it exhibits shrinking, steady, or expanding behavior. Finally, we demonstrate that an $\eta$-Ricci soliton on anti-invariant submanifold of a K-contact manifold becomes $\eta$-Einstein when the potential vector field $V$ is pointwise collinear with $\xi$ with respect to Zamkovoy connection.

# 2 Preliminaries

Let us consider $(2m+1)$-dimensional almost contact metric manifold with almost contact structure $(\phi, \xi, \eta, g)$ defined by a $(1,1)$-tensor field $\phi$, a characteristic vector field $\xi$, a 1-form $\eta$ and a compatible metric g satisfying the condition [3]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X), \tag{7}$$

for all vector fields $X, Y \in \chi(T\tilde{M})$. An almost contact metric manifold is called contact metric manifold if $d\eta(X, Y) = \Phi(X, Y) = g(X, \phi Y)$, where $\Phi$ is the fundamental 2-form of the manifold. If $\xi$ is a Killing vector field, then the manifold $\tilde{M}$ is called a K-contact manifold.
If $\tilde{M}^{2n+1}$ is a K-contact manifold $\tilde{M}$, then the following relations hold [13]:

$$\tilde{\nabla}_X \xi = -\phi X, \tag{8}$$

$$(\tilde{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{9}$$

$$(\tilde{\nabla}_X \eta)Y = -g(\phi X, Y), \tag{10}$$

$$\tilde{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{11}$$

$$S(X, \xi) = 2n\eta(X), \tag{12}$$

$$QX = 2nX, \tag{13}$$

$$r = 2n(2n+1), \tag{14}$$

for every $X, Y \in \chi(T\tilde{M})$, where $\tilde{\nabla}$ represents the Riemannian connection relative to g and $\tilde{R}$, $S$, $Q$ and $r$ represent the Riemannian curvature tensor of type (1, 3), Ricci tensor, Ricci operator and Ricci curvature on $\tilde{M}$, respectively.

For an $n$-dimensional almost contact metric manifold consisting of a (1, 1)-tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian matric $g$ with the

Riemannian connection $\tilde{\nabla}$, the relation for Zamkovoy connection $\nabla^*$ is given by [20]

$$\nabla^*_X Y = \tilde{\nabla}_X Y + (\tilde{\nabla}_X \eta)(Y)\xi - \eta(Y)\tilde{\nabla}_X \xi + \eta(X)\phi Y. \tag{15}$$

Utilizing (10) and (8) in (15), we get

$$\nabla^*_X Y = \tilde{\nabla}_X Y - g(\phi X, Y)\xi + \eta(Y)\phi X + \eta(X)\phi Y. \tag{16}$$

Now, applying (4) in equation (16), we get the expression of Zamkovoy connection on an anti-invariant submanifold $N$ of $\tilde{M}$ as

$$\nabla^*_X Y = \tilde{\nabla}_X Y + \eta(Y)\phi.X + \eta(X)\phi Y. \tag{17}$$

Setting $Y = \xi$ in above equality and using (8), we obtain

$$\nabla^*_X \xi = 0. \tag{18}$$

In view of (9) and (17), we have

$$(\nabla^*_X \phi)Y = g(\phi X, \phi Y)\xi. \tag{19}$$

From (17), we derive

$$(\nabla^*_X \eta)Y = 0. \tag{20}$$

Let $\tilde{R}$ and $R^*$ denote the curvature tensor of $\tilde{\nabla}$ and $\nabla^*$ respectively. By definition

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z. \tag{21}$$

Now, making use of (17) in (21), we get the Riemannian curvature tensor $R^*$ of an anti-invariant submanifold $N$ of $\tilde{M}$ with respect to Zamkovoy connection as

$$R^*(X,Y)Z = \tilde{R}(X,Y)Z + [g(X,Z)\xi - \eta(Z)X]\eta(Y) - [g(Y,Z)\xi - \eta(Z)Y]\eta(X). \tag{22}$$

On contracting equation (22), we obtain Ricci tensor $S^*$ of anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ with respect to Zamkovoy connection of the form

$$S^*(Y,Z) = S(Y,Z) - g(Y,Z) - (2n-1)\eta(Y)\eta(Z). \tag{23}$$

From (23), we obtain

$$Q^*Y = QY - Y - (2n-1)\eta(Y)\xi. \tag{24}$$

Contracting (23) with respect to $Y$ and $Z$, we get

$$r^* = r - 4n, \tag{25}$$

where $r^*$ and $r$ are the scalar curvatures with respect to connections $\nabla^*$ and $\tilde{\nabla}$ respectively.

**Definition 2.1** *A contact metric manifold $\tilde{M}$ is said to be $\eta$-Einstein if $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where $a$ and $b$ are smooth functions on $\tilde{M}$.*

# 3 $\eta$-Ricci soliton on anti-invariant submanifold of a K-contact manifold with respect to Zamkovoy connection

Let $(g, V, \lambda, \mu)$ be an $\eta$-Ricci soliton on anti-invariant submanifold $N$ of $\tilde{M}$ with respect to Zamkovoy connection. From (1), we have

$$(L_V^* g)(Y, Z) + 2S^*(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{26}$$

Taking $V{=}\xi$ in (26), we get

$$(L_\xi^* g)(Y, Z) + 2S^*(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

which implies

$$g(\nabla_Y^* \xi, Z) + g(\nabla_Z^* \xi, Y) + 2S^*(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Utilizing (18) in above equation, we obtain

$$S^*(Y, Z) = -\lambda g(Y, Z) - \mu\eta(Y)\eta(Z). \tag{27}$$

Thus, we state the following theorem

**Theorem 3.1** *Let $(g, \xi, \lambda, \mu)$ be an $\eta$-Ricci soliton on anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ with respect to Zamkovoy connection. Then $N$ become $\eta$-Einstein with respect to Zamkovoy connection.*

Now we study the $\eta$-Ricci soliton of $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$, and we also consider $N$ to be Ricci flat with respect to Zamkovoy connection.

Let $(g, V, \lambda, \mu)$ be an $\eta$-Ricci soliton on the anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$. From equation (1) and for $V = \xi$, we have

$$(L_\xi g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

Applying the property of the Lie derivative, we get

$$g(\nabla_Y \xi, Z) + g(\nabla_Z \xi, Y) + 2S(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0.$$

By utilizing (8) and then applying (4) in the preceding equation and rearranging terms, we derive

$$S(Y, Z) = -\lambda g(Y, Z) - \mu\eta(Y)\eta(Z). \tag{28}$$

Setting $Z = \xi$ in (28), we obtain

$$S(Y, \xi) = -(\lambda + \mu)\eta(Y). \tag{29}$$

Suppose $N$ is Ricci flat with respect to Zamkovoy connection. From equation (23), we deduce

$$S(Y,Z) = g(Y,Z) + (2n-1)\eta(Y)\eta(Z). \tag{30}$$

Putting $Z = \xi$ in (30), we get

$$S(Y,\xi) = 2n\eta(Y). \tag{31}$$

Equating (29) and (31), we obtain

$$\lambda = -2n - \mu. \tag{32}$$

Hence from (32), we state the following theorem as

**Theorem 3.2** *If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ is Ricci flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ is shrinking, steady, or expanding depending on whether $-\mu < 2n$, $-\mu = 2n$, or $-\mu > 2n$, respectively.*

Now from equation (32), we conclude that $\lambda = -2n$, when $\mu = 0$. Therefore, based on Theorem (3.2), we can state the following theorem.

**Corollary 3.3** *If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ is Ricci flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ reduces to a Ricci soliton, and this soliton is shrinking.*

We consider an $\eta$-Ricci soliton on a $(2n+1)$-dimensional concircularly flat anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ with respect to Zamkovoy connection.

Suppose $N$ is concircularly flat with respect to Zamkovoy connection. Then from (2), we obtain

$$R^*(X,Y)Z = \frac{r^*}{2n(2n+1)}(g(Y,Z)X - g(X,Z)Y) \tag{33}$$

Contracting (33) with respect to $W$, taking $X{=}W{=}e_i$, and summing over $i = 1, 2, ...(2n+1)$, we obtain

$$S^*(Y,Z) = \frac{r^*}{(2n+1)}g(Y,Z). \tag{34}$$

Making use of (23) and (25) in (34), we find

$$S(Y,Z) = \frac{r-(2n-1)}{(2n+1)}g(Y,Z) + (2n-1)\eta(Y)\eta(Z). \tag{35}$$

substituting $Z = \xi$ in (35), we obtain

$$S(Y, \xi) = \frac{r + 2n(2n-1)}{(2n+1)}\eta(Y). \tag{36}$$

Now, equating (29) and (36), we arrive at

$$\lambda = -\frac{r + 2n(2n-1)}{(2n+1)} - \mu. \tag{37}$$

Based on the obtained result (37), we state the following theorem

**Theorem 3.4** *If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ is concircularly flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ is shrinking, steady, or expanding depending on whether $-\mu < \frac{r+2n(2n-1)}{(2n+1)}$, $-\mu = \frac{r+2n(2n-1)}{(2n+1)}$, or $-\mu > \frac{r+2n(2n-1)}{(2n+1)}$, respectively.*

From equation (37), we find that $\lambda = -\frac{r+2n(2n-1)}{(2n+1)}$, when $\mu = 0$. Based on this result, we state the following corollary.

**Corollary 3.5** *If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ is concircularly flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ reduces to a Ricci soliton, and this soliton is shrinking.*

Next, we investigate the $\eta$-Ricci soliton on $(2n+1)$-dimensional M-projectively flat anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ with respect to Zamkovoy connection.

Suppose $N$ is M-projectively flat with respect to Zamkovoy connection. Then from equation (3), we obtain

$$R^*(X,Y)Z = \frac{1}{4n}\left[S^*(Y,Z)X - S^*(X,Z)Y\right] + \frac{1}{4n}\left[g(Y,Z)Q^*X - g(X,Z)Q^*Y\right]. \tag{38}$$

Contracting (38) with respect to $W$, taking $X$=$W$=$e_i$ and summing over $i = 1, 2, ...(2n+1)$, we obtain

$$S^*(Y,Z) = \frac{r^*}{(2n+1)}g(Y,Z). \tag{39}$$

which is same as equation (34). Thus, by following the same procedure as in the previous calculation, we obtain the following results

**Theorem 3.6** *If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a Kenmotsu manifold $\tilde{M}$ is M-projectively flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ is shrinking, steady, or expanding depending on whether $-\mu < \frac{r+2n(2n-1)}{(2n+1)}$, $-\mu = \frac{r+2n(2n-1)}{(2n+1)}$, or $-\mu > \frac{r+2n(2n-1)}{(2n+1)}$, respectively.*

**Corollary 3.7** *If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ is M-projectively flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ reduces to a Ricci soliton, and this soliton is shrinking.*

After thoroughly examining the results of Theorems (3.4, 3.6) and corollaries (3.5, 3.7), we arrive at the following two conclusions.
**Conclusion 1.** If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a Kenmotsu manifold $\tilde{M}$ is $(i)$ concircularly flat and $(ii)$ M-projectively flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ is shrinking, steady, or expanding depending on whether $-\mu < \frac{r+2n(2n-1)}{(2n+1)}$, $-\mu = \frac{r+2n(2n-1)}{(2n+1)}$, or $-\mu > \frac{r+2n(2n-1)}{(2n+1)}$, respectively.
**Conclusion 2.** If a $(2n+1)$-dimensional anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ is $(i)$ concircularly flat and $(ii)$ M-projectively flat with respect to Zamkovoy connection, then an $\eta$-Ricci soliton on $N$ reduces to a Ricci soliton, and this soliton is shrinking with $\lambda = -\frac{r+2n(2n-1)}{(2n+1)}$.

Now, Let $(g, V, \lambda, \mu)$ be an $\eta$-Ricci soliton on anti-invariant submanifold $N$ of $\tilde{M}$ admitting Zamkovoy connection such that potential vector field $V$ is pointwise collinear with structure vector field $\xi$ i.e., $V = \alpha\xi$, where $\alpha$ is a function. Then equation (26) holds and using (18), we have

$$(X\alpha)\eta(Y) + (Y\alpha)\eta(X) + 2S^*(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0. \quad (40)$$

Substituting $Y = \xi$ in (40) and employing (12) and (23), we get

$$(X\alpha) + (\xi\alpha)\eta(X) + 2(\lambda+\mu)\eta(X) = 0. \quad (41)$$

Again substituting $X = \xi$ into (41), we obtain

$$(\xi\alpha) = -(\lambda+\mu). \quad (42)$$

By employing (42) in (41), we arrive at

$$(X\alpha) = -(\lambda+\mu)\eta(X). \quad (43)$$

Applying exterior derivative on (43), we get

$$(\lambda+\mu)d\eta = 0. \quad (44)$$

Since $d\eta \neq 0$, from (44), we have

$$\lambda + \mu = 0 \quad \Longrightarrow \lambda = -\mu. \quad (45)$$

Using (45) in (43), we conclude

$$X\alpha = 0,$$

this implies that $\alpha$ is constant. Then equation (40) transforms to

$$S^*(X,Y) = -\lambda g(X,Y) - \mu\eta(X)\eta(Y). \tag{46}$$

In view of (23), equation (46) takes the form

$$S(X,Y) = (1-\lambda)g(X,Y) + (2n-\mu-1)\eta(X)\eta(Y). \tag{47}$$

Hence, from (45) and (47), we state the following theorem

**Theorem 3.8** *If $(g,\xi,\lambda,\mu)$ is an $\eta$-Ricci soliton on anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ admitting Zamkovoy connection, and if $V$ is pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$. Thus, $(N,g)$ is $\eta$-Einstein and the scalars $\lambda$ and $\mu$ are related by (45).*

In particular, for $\mu$=0, (45) yeilds

$$\lambda = 0. \tag{48}$$

Thus we can state

**Theorem 3.9** *If an anti-invariant submanifold $N$ of a K-contact manifold $\tilde{M}$ admitting Zamkovoy connection has a Ricci soliton $(g,\xi,\lambda)$ whose potential vector field $V$ is pointwise collinear with $\xi$, then such soliton is always steady.*

# 4 Open Problem

In this paper, we investigate the properties of $\eta$-Ricci solitons on anti-invariant submanifold of a K-contact metric admitting Zamkovoy connection. Our results elucidate whether the $\eta$-Ricci soliton exhibits shrinking, steady, or expanding behavior. Future research could extend these findings by exploring general connection and $\eta$-Ricci Yamabe solitons, potentially yielding even more intriguing results.

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