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On a new one-parameter arctangent-power integral

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Abstract

This note is devoted to a new special one-parameter arctangent-power integral. For a special value of the parameter, we rely on a result from the literature involving the Catalan constant. The proofs are given in detail. An open problem and some conjectures are also derived.

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1 Introduction

Integration is one of the key concepts in mathematics. New and challenging integrals are still often encountered in modern applications. Unfortunately, traditional methods do not always provide closed-form solutions, and the large panel of existing integrals sometimes does not contain the desired ones. It is therefore important to determine the values of new integrals to improve our understanding of complex mathematical systems, beyond the numerical or approximation techniques commonly used by software (and somewhat imprecise). There is therefore a need for continued research in this area. The following recent references support this claim in an elegant way: [3], [4], [5] and [6].

In particular, there is a long list of integrals of the arctangent-power type in the literature. Most of them are presented in two full sections in [2] (see [2, Sections 4.53 and 4.54]). Nevertheless, there is still room to find new such integrals. In this note, we highlight a new one that depends on an adjustable parameter, which is not so common in this area. To be more precise, we will focus on the following integral:

$$\int_0^1 \frac{1}{1+\theta x^2} \arctan\left[\frac{1-x(2+\theta x)}{1+x(\theta+1-x)}\right] dx, \tag{1}$$

with $\theta \geq -1$. Combining specific integral techniques and some properties of the arctangent function, we show that for $\theta > -1$ it is equal to 0. We also find that the special case $\theta = -1$ corresponds to a known case in the literature, with a more complicated value depending on the Catalan constant. All these aspects are proved and discussed in detail in Section 2. An open problem and some conjectures are formulated in Section 3 and a conclusion is given in Section 4.

2 Results

2.1 Main results

The proposition below is an intermediate result to the proof of our main theorem. It is mainly based on a thorough change of variables.

Proposition 2.1 *For any $\theta > -1$, we have*

$$\int_0^1 \frac{1}{1+\theta x^2} \arctan\left(\frac{1-x}{1+\theta x}\right) dx = \int_0^1 \frac{1}{1+\theta x^2} \arctan(x) dx.$$

Proof of Proposition 2.1. We make the following change of variables: $x = (1-y)/(1+\theta y)$, in such a way that

$$\frac{1-x}{1+\theta x} = \frac{1-(1-y)/(1+\theta y)}{1+\theta(1-y)/(1+\theta y)} = \frac{(1+\theta y)-(1-y)}{1+\theta y+\theta(1-y)} = \frac{(\theta+1)y}{\theta+1} = y,$$

$$dx = \frac{-(1+\theta y)-\theta(1-y)}{(1+\theta y)^2} dy = -\frac{\theta+1}{(1+\theta y)^2} dy,$$

noticing that $x = 0$ is equivalent to $y = 1$, $x = 1$ is equivalent to $y = 0$, and

$$(1+\theta y)^2 + \theta(1-y)^2 = 1 + 2\theta y + \theta^2 y^2 + \theta - 2\theta y + \theta y^2 = (\theta+1)(1+\theta y^2).$$

We thus obtain

$$\begin{aligned}
&\int_0^1 \frac{1}{1+\theta x^2} \arctan\left(\frac{1-x}{1+\theta x}\right) dx \\
&= \int_1^0 \frac{1}{1+\theta(1-y)^2/(1+\theta y)^2} \arctan(y) \left[-\frac{\theta+1}{(1+\theta y)^2} dy\right] \\
&= (\theta+1) \int_0^1 \frac{1}{(1+\theta y)^2+\theta(1-y)^2} \arctan(y) dy \\
&= \int_0^1 \frac{1}{1+\theta y^2} \arctan(y) dy.
\end{aligned}$$

By standardizing the notation, we obtain the desired integral result. The proof of Proposition 2.1 is concluded. □

Based on this proposition, the following theorem is derived. It gives the exact value of the integral in Equation (1) for $\theta > -1$.

Theorem 2.2 *For any $\theta > -1$, we have*

$$\int_0^1 \frac{1}{1+\theta x^2} \arctan\left[\frac{1-x(2+\theta x)}{1+x(\theta+1-x)}\right] dx = 0.$$

Proof of Theorem 2.2. The difference of the two integrals in Proposition 2.1 gives

$$\int_0^1 \frac{1}{1+\theta x^2} \arctan\left(\frac{1-x}{1+\theta x}\right) dx - \int_0^1 \frac{1}{1+\theta x^2} \arctan(x) dx = 0,$$

which can also be written as

$$\int_0^1 \frac{1}{1+\theta x^2} \left[\arctan\left(\frac{1-x}{1+\theta x}\right) - \arctan(x)\right] dx = 0. \tag{2}$$

We now recall that, for any variables u and v such that $uv > -1$, the following arctangent summation holds:

$$\arctan(u) - \arctan(v) = \arctan\left(\frac{u-v}{1+uv}\right). \tag{3}$$

Notice that, for any $\theta > -1$ and $x \in (0, 1)$, if we set $u = (1-x)/(1+\theta x)$ and $v = x$, we have

$$uv = x\frac{1-x}{1+\theta x} \geq \frac{x(1-x)}{\min(1, 1+\theta)} \geq 0 > -1.$$

It follows from Equations (2) and (3) under this configuration that

$$\int_0^1 \frac{1}{1+\theta x^2} \arctan\left[\frac{(1-x)/(1+\theta x)-x}{1+x(1-x)/(1+\theta x)}\right] dx = 0,$$

which can also be written as follows, after some basic manipulations for the ratio term into the arctangent function:

$$\int_0^1 \frac{1}{1+\theta x^2} \arctan\left[\frac{1-x(2+\theta x)}{1+x(\theta+1-x)}\right] dx = 0.$$

This ends the proof of Theorem 2.2. □

To the best of our knowledge, this one-parameter arctangent-power integral result is new in the literature, as claimed in the introduction.

The special case $\theta = -1$ requires different techniques. More specifically, we have

$$\int_0^1 \frac{1}{1-x^2} \arctan\left[\frac{1-x(2-x)}{1+x(-1+1-x)}\right] dx = \int_0^1 \frac{1}{1-x^2} \arctan\left[\frac{(1-x)^2}{1-x^2}\right] dx$$
$$= \int_0^1 \frac{1}{1-x^2} \arctan\left(\frac{1-x}{1+x}\right) dx.$$

This special integral already exists in the literature; we have identified it as a part of an existing proof, in [1, Part of the proof of (3), page 9]. Based on this, we have

$$\int_0^1 \frac{1}{1-x^2} \arctan\left(\frac{1-x}{1+x}\right) dx = \frac{1}{2}G,$$

where G denotes the famous Catalan constant defined by

$$G = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^2} \approx 0.915965594\ldots$$

In a sense, Theorem 2.2 provides the opposite case, i.e., with $\theta > -1$, with a surprisingly uniform value of 0.

2.2 Complements

We end this section with a general integral formula. It is related to Proposition 2.1 and Theorem 2.2, using an arbitrary function.

Proposition 2.3 *For any $\theta > -1$ and any function $f : \mathbb{R} \mapsto \mathbb{R}$, we have*

$$\int_0^1 \frac{1}{1+\theta x^2} \left[f\left(\frac{1-x}{1+\theta x}\right) - f(x)\right] dx = 0,$$

provided that the integral exists.

Proof of Proposition 2.3. If we follow line by line the proof of Proposition 2.1, and simply replace $\arctan(x)$ by $f(x)$, we get

$$\int_0^1 \frac{1}{1+\theta x^2} f\left(\frac{1-x}{1+\theta x}\right) dx = \int_0^1 \frac{1}{1+\theta x^2} f(x) dx.$$

This imples that

$$\int_0^1 \frac{1}{1+\theta x^2} f\left(\frac{1-x}{1+\theta x}\right) dx - \int_0^1 \frac{1}{1+\theta x^2} f(x) dx = 0$$

and

$$\int_0^1 \frac{1}{1+\theta x^2} \left[f\left(\frac{1-x}{1+\theta x}\right) - f(x)\right] dx = 0.$$

This ends the proof of Proposition 2.3. □

Some basic examples of Proposition 2.3 are now presented.

- If we take $f(x) = \arctan(x)$ in Proposition 2.3, we get Proposition 2.1.
- We can also consider $f(x) = \log(x)$. For any $\theta > -1$, we then have

$$\int_0^1 \frac{1}{1+\theta x^2} \log\left[\frac{1-x}{x(1+\theta x)}\right] dx = \int_0^1 \frac{1}{1+\theta x^2} \left[\log\left(\frac{1-x}{1+\theta x}\right) - \log(x)\right] dx = 0.$$

  This integral is not given in [2]. Note that, for the special case $\theta = -1$, we can prove that it is still equal to 0.

- Another simple example comes from the choice of $f(x) = x$. For any $\theta > -1$, we have

$$\int_0^1 \frac{1-2x-\theta x^2}{(1+\theta x^2)(1+\theta x)} dx = \int_0^1 \frac{1}{1+\theta x^2} \left(\frac{1-x}{1+\theta x} - x\right) dx = 0.$$

  Note that, for the special case $\theta = -1$, simplifications and a direct primitive calculation give

$$\int_0^1 \frac{1-2x+x^2}{(1-x^2)(1-x)} dx = \int_0^1 \frac{1}{1+x} dx = \log(2).$$

- Trigonometric integral types can be considered too. Selecting $f(x) = \sin(x)$ and using the formula $\sin(u) - \sin(v) = 2\sin[(u-v)/2]\cos[(u+v)/2]$, we get

$$\int_0^1 \frac{1}{1+\theta x^2} \sin\left(\frac{1-2x-\theta x^2}{1+\theta x}\right) \cos\left(\frac{1+\theta x^2}{1+\theta x}\right) dx$$
$$= \frac{1}{2}\int_0^1 \frac{1}{1+\theta x^2}\left[\sin\left(\frac{1-x}{1+\theta x}\right) - \sin(x)\right] dx = 0.$$

- Similarly, selecting $f(x) = \cos(x)$ and using the formula $\cos(u)-\cos(v) = -2\sin[(u-v)/2]\sin[(u+v)/2]$, we get

$$\int_0^1 \frac{1}{1+\theta x^2} \sin\left(\frac{1-2x-\theta x^2}{1+\theta x}\right) \sin\left(\frac{1+\theta x^2}{1+\theta x}\right) dx$$
$$= -\frac{1}{2}\int_0^1 \frac{1}{1+\theta x^2}\left[\cos\left(\frac{1-x}{1+\theta x}\right) - \cos(x)\right] dx = 0.$$

Many more examples can be given on the basis of Proposition 2.3.

A direct consequence of this proposition, which also justifies the presence of the parameter α, is the following result: assuming the Leibniz integral rule, for any positive integer m, we have

$$\int_0^1 \frac{\partial^m}{\partial \theta^m}\left\{\frac{1}{1+\theta x^2}\left[f\left(\frac{1-x}{1+\theta x}\right) - f(x)\right]\right\} dx$$
$$\frac{\partial^m}{\partial \theta^m}\left\{\int_0^1 \frac{1}{1+\theta x^2}\left[f\left(\frac{1-x}{1+\theta x}\right) - f(x)\right] dx\right\} = \frac{\partial^m}{\partial \theta^m} 0 = 0.$$

This opens up the construction of a wide range of new integrals defined on the interval $(0,1)$ and equal to 0.

3 An open problem and conjectures

3.1 An open problem

The following open problem, formulated as a question, comes from Theorem 2.2: What is the value of the following improper integral:

$$\int_1^{+\infty} \frac{1}{1+\theta x^2} \arctan\left[\frac{1-x(2+\theta x)}{1+x(\theta+1-x)}\right] dx?$$

(The main change is in the integration interval considered). No solution is actually found. Note that, given the integral in Theorem 2.2 and the Chasles relation, the challenging integral can also be expressed as follows, with the same theoretical value:

$$\int_0^{+\infty} \frac{1}{1+\theta x^2} \arctan\left[\frac{1-x(2+\theta x)}{1+x(\theta+1-x)}\right] dx.$$

3.2 Conjectures

During our complementary research, the following conjectures also appear: For any $\theta > -1$ and any positive integers k and ℓ, we have

$$\int_0^1 \frac{1}{1+\theta x^2} \left[\sin\left(\frac{1-2x-\theta x^2}{1+\theta x}\right)\right]^{2k+1} \left[\tan\left(\frac{1+\theta x^2}{1+\theta x}\right)\right]^{\ell} dx = 0,$$

$$\int_0^1 \frac{1}{1+\theta x^2} \left[\sin\left(\frac{1-2x-\theta x^2}{1+\theta x}\right)\right]^{2k+1} \left[\operatorname{cotan}\left(\frac{1+\theta x^2}{1+\theta x}\right)\right]^{\ell} dx = 0,$$

$$\int_0^1 \frac{1}{1+\theta x^2} \left[\tan\left(\frac{1-2x-\theta x^2}{1+\theta x}\right)\right]^{2k+1} \left[\tan\left(\frac{1+\theta x^2}{1+\theta x}\right)\right]^{2\ell+1} dx = 0$$

and

$$\int_0^1 \frac{1}{1+\theta x^2} \left[\tan\left(\frac{1-2x-\theta x^2}{1+\theta x}\right)\right]^{2k+1} \left[\operatorname{cotan}\left(\frac{1+\theta x^2}{1+\theta x}\right)\right]^{\ell} dx = 0.$$

The rigorous proofs of these results have yet to be given. Note that we also conjecture that the above integrals are still valid if we replace the trigonometric functions involved by their hyperbolic analogues, i.e., sinh instead of sin, tanh instead of tan, and cotanh instead of cotan.

4 Conclusion

In this note, we have given the exact value of a new special one-parameter arctangent-power integral not presented in [2]. The value obtained is 0, except for a special value of the parameter involved. The proof is based on a thorough change of variables and the use of the summation property of the arctangent function. An open problem derived from the main theorem is given, which poses a challenging calculation of a derived improper integral.

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