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On geometric properties of generalized subclass of analytic functions

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Abstract

In this work, we introduce a new class of analytic functions defined by a generalized operator denoted as $\mathcal{B}_\sigma^n(\alpha, \beta)$ and obtain the following properties namely: inclusion property, integral representations, sufficient univalency condition of the new class, coefficient inequalities and Fekete-Szego inequalities.

Keywords: *Analytic and univalent functions, integral operator, Salagean derivative and its anti-derivative operators.*

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1 Introduction

The study of normalized univalent function f of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, in the unit disk, has its beginning from the Basilevic map which was introduced by in 1955 and define as follows

$$f(z) = \left[\frac{\alpha}{1+\beta^2}\int_0^z [p(t) - i\beta]t^{-(1+\frac{i\alpha\beta}{1+\beta^2})}g(t)^{(\frac{\alpha}{1+\beta^2})}\right]. \tag{1}$$

where $p \in P$ refers to as functions with positive real part and $g(z) = z + b_2z^2 + \cdots$ is starlike with the parameters $\alpha > 0$ and β are real and all powers mean principal determinations only.

The study of this Bazilevic map gave rise to introduction of many of its subclasses by many researchers in the theory of geometric functions.
Abdulhalim (1992), further generalized the classes of functions with

$$\frac{\mathcal{D}^n f(z)^\alpha}{z^\alpha}, \quad n \in \mathcal{N} \tag{2}$$

having positive real parts in U.
Opoola (1994), generalized the class of analytic functions, that for a real number β, $(0 \leq \beta < 1)$, then

$$Re\left[\frac{\mathcal{D}^n f(z)^\alpha}{\alpha^n z^\alpha}\right] > \beta, \quad z \in U. \tag{3}$$

He denoted the generalised class by $T_n^\alpha(\beta)$ and Babalola (2005) investigated the class $T_n^\alpha(\beta)$ and determined the coefficient bounds, growth, distortion, radius and some transformations of functions in the class.
Salagean (1983), used the operator to generalized the concept of starlikeness and convexity of the function denoted as the class $S_n(\beta)$ with the geometric conditions

$$Re\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} > \beta.$$

Tuan (1978), obtained the largest disk mapped by the functions in the class onto a starlike domain and also obtained the radius of starlikeness of this class of functions. In this work, we consider the differential operator D^n such that $n \in \mathcal{N} \cup \{0\}$, which is the salagean differential operator defined as $D^n f(z) = D(D^{n-1}f(z)) = z[D^{n-1}f(z)]$ with $D^0 f(z) = f(z)$, introduced in (1983) and the integral operator of one parameter denoted as I^σ define as $I^\sigma = f(z) = \frac{2^\sigma}{z\Gamma\sigma}\int_0^z (log\frac{z}{t})^{\sigma-1} f(t)dt$ introduced by Jung-Kim-Srivastava in (1993) which was used to generalized classes of analytic functions.
Using the salagean differential operator on $f(z)^\alpha$ define as :

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} A_k(\alpha) a_k z^{\alpha+k-1}$$

where $A_k(\alpha)$ depends on the coefficient of a_k.
We have

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} A_k(\alpha)(\alpha + k - 1)^n z^{\alpha+k-1}.$$

Also, using one-parameter Jung-Kim-Srivastava integral operator on $f(z)^\zeta$ and by normalization, we have

$$I^\sigma f(z)^\alpha = \frac{(\alpha+1)^\sigma}{z\Gamma\sigma}\int_0^z (\log\frac{z}{t})^{\sigma-1} f(t)^\alpha dt = z^\alpha + \sum_{k=2}^{\infty}\left(\frac{\alpha+1}{\alpha+k}\right)^\sigma A_k(\alpha) z^{\alpha+k-1}.$$

Therefore, we have

$$D^n(I^\sigma f(z)^\alpha) = \alpha^n z^\alpha + \sum_{k=2}^{\infty} \left(\frac{\alpha+1}{\alpha+k}\right)^\sigma (\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1} = I^\sigma(D^n f(z)^\alpha) = \mathcal{L}_\sigma^n f(z)^\alpha.$$

Remark 1.1 *We have that* $\mathcal{L}_\sigma^0 f(z)^\alpha = I^\sigma f(z)^\alpha$, $\mathcal{L}_0^n f(z)^\alpha = D^n f(z)^\alpha$, *so that* $\mathcal{L}_0^0 f(z)^\alpha = f(z)^\alpha$.

The salagean integral operator is define as

$$I_n f(z) = I_n(I_{n-1} f(t)) = \int_0^z \frac{f(t)}{t} dt.$$

Then

$$I_n f(z)^\alpha = \frac{z^\alpha}{\alpha^n} + \sum_{k=2}^{\infty} \frac{A_k(\alpha)}{(\alpha+k-1)^n} z^{\alpha+k-1}.$$

Considering the method adopted by Babalola (2012), we have the inverse of the integral operator $I^\sigma f(z)$, denoted as $I^{-\sigma} f(z)$ defined as follows:

$$I^{-\sigma} f(z) = \frac{2^{-\sigma}}{z\Gamma(-\sigma)} \int_0^z (log\frac{z}{t})^{-\sigma-1} f(t) dt.$$

Then, we have

$$I^{-\sigma} f(z)^\alpha = \frac{(\alpha+1)^{-\sigma}}{z\Gamma(-\sigma)} \int_0^z (log\frac{z}{t})^{-\sigma-1} f(t)^\alpha dt = z^\alpha + \sum_{k=2}^{\infty} \left(\frac{\alpha+k}{\alpha+1}\right)^\sigma A_k(\alpha) z^{\alpha+k-1}.$$

So that

$$I_n(I^{-\sigma} f(z)^\alpha) = \frac{z^\alpha}{\alpha^n} + \sum_{k=2}^{\infty} \left(\frac{\alpha+k}{\alpha+1}\right)^\sigma \frac{A_k(\alpha)}{(\alpha+k-1)^n} z^{\alpha+k-1} = \mathcal{J}_\sigma^n f(z)^\alpha.$$

Remark 1.2 *We have* $\mathcal{J}_\sigma^0 f(z)^\alpha = I^{-\sigma} f(z)^\alpha$, $\mathcal{J}_0^n f(z)^\alpha = I_n f(z)^\alpha$, $\mathcal{J}_0^0 f(z)^\alpha = f(z)^\alpha$ *so that* $\mathcal{L}_\sigma^n(\mathcal{J}_\sigma^n f(z)^\alpha) = \mathcal{J}_\sigma^n(\mathcal{L}_\sigma^n f(z)^\alpha) = f(z)^\alpha$.

Using the operator $\mathcal{L}_\sigma^n$, we define the class $\mathcal{B}_\sigma^n(\alpha, \beta)$ as follows:

Definition 1.3 *Let* $f \in A$, *then* $f(z)$ *is said to be in the class* $\mathcal{B}_\sigma^n(\alpha, \beta)$, *if it satisfies the geometric condition*

$$\frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta, 0 \le \beta < 1, \alpha > 0. \quad (4)$$

Remark 1.4 *(i) If $\sigma = 0$, we have the class $T_\alpha^n(\beta)$ introduced by Opoola in (1994) and studied by babalola, (2005).*
(ii) If $n = 0$, we have the class $\frac{I^\sigma f(z)^\alpha}{z^\alpha} > \beta$ introduced and studied by Liu and Owa in (1998).
(iii) If $n = 0$, $\sigma = 0$, we have the class $\frac{f(z)^\alpha}{z^\alpha} > \beta$.
(iv) If $n = 0$, $\sigma = 0$ and $\alpha = 1$, we have the class $\frac{f(z)}{z} > \beta$.
(v) If $n = 0$, $\sigma = 0$, $\beta = 0$ and $\alpha = 1$, we have the class $\frac{f(z)}{z} > 0$ which was introduced and studied by Yamaguchi in (1966).

Our objectives in this work is to obtain the following properties namely: inclusion property, integral representations, sufficient univalency condition of the new class, coefficient inequalities and Fekete-Szego inequalities.
In the next section, we state lemmas to obtain our man results.

2 Preliminaries Lemmas

Lemma 2.1 *[4] Let $p(z)$ be holomorphic in E with $p(0) = 1$. Suppose that*

$$Re\left(1 + \frac{zp^{'}(z)}{p(z)}\right) > \frac{3\beta - 1}{2\beta}.$$

Then

$$Rep(z) > 2^{1-\frac{1}{\beta}}, \frac{1}{2} \leq \beta < 1, z \in U. \tag{5}$$

and the constant $2^{1-\frac{1}{\beta}}$ is the best possible.

Lemma 2.2 *[5] Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\Phi(u, v)$ a complex valued function satisfying*
(i) $\Phi(u, v)$ is continuous in a domain Ω of C^2.
(ii) $(1, 0) \in \Omega$ and $Re\Phi(1, 0) > 0$.
(iii) $Re\Phi(\beta + (1 - \beta)u_2 i, v_1) \leq \beta$ when $(\beta + (1 - \beta)u_2 i, v_1) \in \Omega$
If $p \in P$ such that $(p(z), zp^{'}(z)) \in \Omega$ and $Re(p(z), zp^{'}(z)) > \beta$ for $z \in \mathcal{U}$. Then $Rep(z) > \beta$ in $\mathcal{U}$.

Lemma 2.3 *[9] Let $p \in P$. where $p(z) = 1 + c_1 z + p_2 z^2 + \cdots$, then*

$$|p_k| \leq 2, k = 1, 2, 3, \cdots . \tag{6}$$

Lemma 2.4 *[6] Let $p \in P$. then for any real or complex number μ, we have sharp inequalities*

$$\left|p_2 - \mu\frac{p_1^2}{2}\right| \leq 2\max\{1, |1 - \mu|\}. \tag{7}$$

3 Main results

Theorem 3.1 $\mathcal{B}_\sigma^{n+1}(\alpha,\beta) \subset \mathcal{B}_\sigma^n(\alpha,\beta)$, $\alpha > 0$, $0 \leq \beta < 1$.

Proof Since $f \in \mathcal{B}_\sigma^n(\alpha,\beta)$, there exist $p_\beta \in P$ such that

$$p(z) = \frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha}. \tag{8}$$

So that

$$zp'(z) = \frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\alpha^n z^\alpha} - \alpha \frac{z^\alpha \mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^{2\alpha}}.$$

Then, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\mathcal{L}_\sigma^n f(z)^\alpha} - \alpha.$$

and

$$\frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\mathcal{L}_\sigma^n f(z)^\alpha} = \frac{zp'(z)}{p(z)} + \alpha$$

Multiply through by $\frac{p(z)}{\alpha}$, we obtain

$$\frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha} = p(z) + \frac{zp'(z)}{\alpha}$$

$$Re\left(\frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\alpha^{n+1} z^\alpha}\right) = Re\left(p(z) + \frac{zp'(z)}{\alpha}\right) > \beta. \tag{9}$$

We define

$$\Psi(u,v) = u + \frac{v}{\alpha}$$

on the domain $\Omega = \mathcal{C}^2$. Then $\Psi(u,v)$ satisfies the condition (i) and (ii) of Lemma 2.2
Also

$$Re\Psi(\beta + (1-\beta)u_2 i, v_1) = \beta + \frac{v_1}{\alpha} < \beta$$

whenever $v_1 \leq -\frac{(1-\beta)(1+u_2^2)}{2}$. Therefore , $\Psi(u,v)$ satisfies al the conditions of the Lemma 2.2 and so

$$\frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha} > \beta.$$

Corollary 3.2 *For $n \geq 1$, $\sigma = 0$ and $\alpha = 1$, the class $\mathcal{B}_\sigma^n(\alpha,\beta)$ consists of univalent functions only in $\mathcal{U}$. In particular the class of function defined by $Ref'(z) > \beta$.*

Theorem 3.3 *Let $f \in \mathcal{B}_\sigma^n(\alpha,\beta)$. Then $f(z)$ has the integral representation*

$$f(z) = \left(\mathcal{J}_\sigma^n\left(\alpha^n z^\alpha p(t)\right)\right)^{\frac{1}{\alpha}}.$$

Proof. Since $f \in \mathcal{B}_\sigma^n(\alpha,\beta)$, there exist $P_\beta \in P$ such that

$$\frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha} = p(z). \tag{10}$$

Since the integral representation of $\mathcal{L}_\sigma^n$ is $\mathcal{J}_\sigma^n$.
Then

$$\mathcal{L}_\sigma^n f(z)^\alpha = \alpha^n z^\alpha p(z)$$

and

$$f(z)^\alpha = \mathcal{J}_\sigma^n(\alpha^n z^\alpha p(z)).$$

So that

$$f(z) = (\mathcal{J}_\sigma^n(\alpha^n z^\alpha p(z)))^{\frac{1}{\alpha}}.$$

Corollary 3.4 *For $\sigma = 0$. Then*

$$f(z) = (I_n(\alpha^n z^\alpha p(t)dt))^{\frac{1}{\alpha}}$$

. *as*

$$I_n = I(I_{n-1}f(z)) = \int \frac{I_{n-1}f(t)}{t}dt$$

Corollary 3.5 *For $n = 0$, Then*

$$f(z) = (I^{-\sigma}(\alpha^n z^\alpha p(t)dt))^{\frac{1}{\alpha}}$$

as

$$I^{-\sigma}f(z) = \frac{2^{-\sigma}}{z\Gamma(-\sigma)}\int_0^z (log\frac{z}{t})^{-\sigma-1}f(t)dt$$

Theorem 3.6 *If $f \in A$ satisfies the condition*

$$Re\left(\frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\mathcal{L}_\sigma^n f(z)^\alpha}\right) > \frac{\beta + 2\alpha\beta - 1}{2\beta}. \tag{11}$$

Then $\frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha} > 2^{1-\frac{1}{\beta}}$, $\frac{1}{2} \leq \beta < 1$, $z \in \mathcal{U}$

Proof. Since $f \in \mathcal{B}_\sigma^n(\alpha, \beta)$, there exist $p_\beta \in P$ such that

$$p(z) = \frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha}. \tag{12}$$

So that

$$zp'(z) = \frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\alpha^n z^\alpha} - \alpha\frac{z^\alpha \mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^{2\alpha}}.$$

Then, we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\mathcal{L}_\sigma^n f(z)^\alpha} - \alpha.$$

and

$$\left(1 + \frac{zp'(z)}{p(z)}\right) = Re\left(\frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\mathcal{L}_\sigma^n f(z)^\alpha} + 1 - \alpha\right) > \frac{3\beta - 1}{2\beta}.$$

which is equivalent to

$$\left(\frac{\mathcal{L}_\sigma^{n+1} f(z)^\alpha}{\mathcal{L}_\sigma^n f(z)^\alpha}\right) > \frac{\beta + 2\beta\alpha - 1}{2\beta}.$$

By Lemma 2.1, $Rep(z) > 2^{1-\frac{1}{\beta}}$, $\frac{1}{2} \leq \beta < 1$ and the proof completes.

Corollary 3.7 *If $f \in A$ satisfies the condition (4.4), then $f \in B_\sigma^n(\alpha, 2^{1-\frac{1}{\beta}})$.*

Corollary 3.8 *If $n = 0$, $\sigma = 0$, and $\alpha = 1$ we have*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{3\beta - 1}{2\beta}.$$

Then

$$Re\frac{f(z)}{z} > 2^{1-\frac{1}{\beta}}.$$

Corollary 3.9 *If $n = 1$, $\sigma = 0$, $\alpha = 1$, we have*

$$Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \frac{3\beta - 1}{2\beta}$$

. *Then*

$$Ref'(z) > 2^{1-\frac{1}{\beta}}.$$

Corollary 3.10 *If $n=0$, $\sigma=0$, $\alpha=1$ and $\beta=1/2$ we have*

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2}.$$

Then

$$Re\frac{f(z)}{z} > \frac{1}{2}.$$

Corollary 3.11 *If $n=1$, $\sigma=0$, $\alpha=1$ and $\beta=1/2$ we have*

$$Re\left(\frac{zf''(z)}{f'(z)}+1\right) > \frac{1}{2}$$

. *Then*

$$Ref'(z) > \frac{1}{2}.$$

Theorem 3.12 *Let $f \in B_\sigma^n(\alpha,\beta)$, then*

$$|a_2| \leq \tfrac{2\alpha^{n-1}(1-\beta)(\alpha+1)^\sigma}{(\alpha+1)^n 2^\sigma}.$$

$$|a_3| \leq \frac{2\alpha^{n-1}(\alpha+2)^\sigma(1-\beta)}{(\alpha+2)^n 2^\sigma} max\{1, |1-\mathbf{M_1}|\} \tag{13}$$

where

$$\mathbf{M_1} = \frac{2\alpha^{n-1}(1-\alpha)2^\sigma}{(\alpha+1)^n(\alpha+2)^\sigma}.$$

Proof. Since $f \in \mathcal{B}_\sigma^n(\alpha,\beta)$, then there exists $p \in P_\beta$ such that

$$\frac{\mathcal{L}_\sigma^n f(z)^\alpha}{\alpha^n z^\alpha} = 1 + (1-\beta)c_1 z + (1-\beta)c_2 z^2 + (1-\beta)c_3 c^3 + \cdots \tag{14}$$

$$\mathcal{L}_{\sigma,\gamma}^n f(z)^\alpha = \alpha^n z^\alpha + \alpha^n(1-\beta)c_1 z^{\alpha+1} + \alpha^n(1-\beta)c_2 z^{\alpha+2} + \alpha^n(1-\beta)c_3 z^{\alpha+3} + \alpha^{n-1}(1-\beta)c_4 z^{\alpha+4} + \cdots \tag{15}$$

Using the anti-derivative of the operator $\mathcal{L}_\sigma^n$ denoted as $\mathcal{J}_\sigma^n$, we have that

$$f(z)^\alpha = z^\alpha + \tfrac{\alpha^n(1-\beta)}{(\alpha+1)^n}\left(\tfrac{\alpha+1}{2}\right)^\sigma c_1 z^{\alpha+1} + \tfrac{\alpha^n(1-\beta)}{(\alpha+2)^n}\left(\tfrac{\alpha+2}{2}\right)^\sigma c_2 z^{\alpha+2} \tag{16}$$

$$+\tfrac{\alpha^n(1-\beta)}{(\alpha+3)^n}\left(\tfrac{\alpha+3}{3}\right)^\sigma c_3 z^{\alpha+2} + \tfrac{\alpha^n(1-\beta)}{(\alpha+4)^n}\left(\tfrac{\alpha+4}{2}\right)^\sigma c_4 z^{\alpha+4}\cdots \tag{17}$$

Given that

$$f(z)^\alpha = z^\alpha + \alpha a_2 z^{\alpha+1} + \left(\alpha a_3 + \tfrac{\alpha(\alpha-1)}{2}a_2^2\right) z^{\alpha+2} + \left(\alpha a_4 + \alpha(\alpha-1)a_2 a_3 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{6}a_2^3\right) z^{\alpha+3} \tag{18}$$

$$+\left(\alpha a_5 + \alpha(\alpha-1)a_2 a_4 + \tfrac{\alpha(\alpha-1)}{2}a_3^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{2}a_2^2 a_3 + \tfrac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{12}a_2^4\right) z^{\alpha+4} + \cdots \tag{19}$$

By comparing the coefficient , we have

$$a_2 = \frac{\alpha^{n-1}}{(\alpha+1)^n}\left(\frac{\alpha+1}{2}\right)^\sigma c_1$$

By Lemma 2.3, we obtained the bound of a_2, also

$$a_3 = \frac{\alpha^{n-1}(\alpha+2)^\sigma(1-\beta)}{(\alpha+2)^n 2^\sigma}\left[c_2 - \frac{\alpha^{n-1}(\alpha-1)(\alpha+1)^{2\sigma}(\alpha+2)^n}{(\alpha+1)^{2n}2^\sigma}\frac{c_1^2}{2}\right] \tag{20}$$

By Lemma 2.4 and with

$$\rho = \frac{\alpha^{n-1}(\alpha-1)(1-\beta)(\alpha+1)^{2\sigma}(\alpha+2)^n}{(\alpha+1)^{2n}2^\sigma},$$

we obtained the bound on the third coefficient of these function and the proof completes.

Theorem 3.13 *Let $f \in B_\sigma^n(\alpha,\beta)$. Then*

$$|a_3 - \rho a_2^2| \leq \frac{\alpha^{n-1}(1-\beta)(\alpha+\gamma)^\sigma}{(\alpha+2)^n(\alpha+2)^\sigma} max\{1, |1-\mathbf{M_2}|\} \tag{21}$$

where

$$\mathbf{M_2} = \frac{2(\alpha+1)^{2n}(\alpha+1)^{2\sigma} + (1+2\rho-\alpha)\alpha^{n-1}(\alpha+1)^\sigma(\alpha+2)^n(\alpha+2)^\sigma}{(\alpha+1)^{2n}(\alpha+1)^{2\sigma}}$$

Proof. From the computation and by comparing coefficient with respect to z, then

$$a_2 = \frac{\alpha^{n-1}(1-\beta)}{(\alpha+1)^n}\left(\frac{\alpha+1}{2}\right)^\sigma c_1 \tag{22}$$

and

$$a_3 = \frac{\alpha^{n-1}(\alpha+2)^\sigma c_2}{(\alpha+2)^n 2^\sigma} - \frac{(\alpha-1)\alpha^{2n-2}(\alpha+1)^{2\sigma}}{(\alpha+1)^{2n}2^{2\sigma}}\frac{c_1^2}{2} \tag{23}$$

Hence

$$|a_3 - \rho a_2^2| = \frac{\alpha^{n-1}(\alpha+2)^\sigma}{(\alpha+2)^n 2^\sigma}c_2 - \frac{(\alpha-1+2\rho)(\alpha+2)^n\alpha^{n-1}(\alpha+1)^\sigma(\alpha+2)^\sigma}{(\alpha+1)^{2n}(\alpha+1)^{2\sigma}}\frac{c_1^2}{2}, \tag{24}$$

by lemma 2.4 we have the required inequality.

4 Open Problem

Further study of this class of function $\mathcal{B}_\sigma^n(\alpha,\beta)$ are ongoing by the authors on the following properties namely: growth, covering, distortion,closure under certain integral transformation.

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References

[1] S. Abdulhalim, *On a class of analytic functions involving the Salagean differential operator*, Tamkang Journal of mathematics 23(1) 1992, pages 51 – 58.

[2] K. O. Babalola, *Some new results on a certain family of analytic functions Defined by the salagean Derivative* Unpublished Doctoral Thesis University of Ilorin.

[3] K. O. Babalola, *new Subclasses od analytic and univalent function involving certain Convolution Operator.*. Mathematica (Cluij), Tom 74(2) 2008, 111-118

[4] K. O. Babalola, On *λ-Pseudo-Starlike functions*, Journal of Classical Analysis, 3, (2) 2013, 137-147.

[5] K. O. Babalola, and T. O. Opoolo, *Iterated integral transforms of Caratheodory functions and their applications to analytic and Univalent functions*, Tamkang Journal of Mathematics, 37(4) 2006, 355-366.

[6] K. O. Babalola and T. O. Opoola, *On the coefficients of a certain class of analytic functions*, Advances in Inequalities for Series, 2008, 1-13.

[7] K. O. Babalola, *Subclasses od analytic and univalent function Defined by Inverse Integral Operator.* I. Mathematica (Cluij), Tom 51(74)(2) 2012, 255-264.

[8] K. O. Babalola, *Some Applications of a Lemma Concerning Analytic Functions with Positive Real Parts in the Unit Disk.* International Journal of Mathematics and Computer Science, no. 4 2007, 361-369.

[9] C. Caratheodŏry, *Theory of functions of a complex variable.* II. Chelsea Publishing Co. New York 1960.

[10] I. B. Jung, Y. C. Kim, Y. & H. M. Strivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators*, J. Math. Anal. Appl., 176 1993, 138–147.

[11] 1 P. Koebe, *Uber die uniformisiesrung beliebiger analytischeer kurven, Nach.* Ges.Wiss.Gottingen, 1907, 191-210

[12] L. J. Liu and S. Owa, *Properties of the salagean operator.* Georgian Mathematics Journal 5(4) 1998, 361-366

[13] Macgregor, T. H. T *The radius of convexity for starlike functions of order 1/2.* Proceedings of American Mathematical Society. 14 1963, 71-76.

[14] T. O. Opoola, *On a new subclass of Univalent functions,* mathematica (cluj) 36, 59 (2) 1994, pages 195-200.