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Uniqueness of Differential Polynomials sharing two values

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Abstract

In this paper, with the help of the notion of multiplicity, we study the uniqueness of differential polynomials that share two values and obtain some results that improve the results of Wang and Gao [8], Yang and Hau [11], and Fang and Qiu [2]. We also solve an open problem posed by Harina P. Waghamore and Ramya Maligi [9].

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1 Introduction and Main Results

Throughout this note, a meromorphic function means meromorphic in the open complex plane $\mathbb{C}$. We shall use the standard notations of value distribution theory of meromorphic functions such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ and so on that can be found, for instance, in [3]. Let f and g be non-constant meromorphic functions, $a \in \mathbb{C}$. We say that f and g share the value a CM (Counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities. Let k be a positive integer, we denote by $N_{k)}\left(r, \frac{1}{f-a}\right)$

the counting function of the roots of $f - a$ with multiplicity $\leq k$ and by $N_{(k+1}\left(r, \frac{1}{f-a}\right)$ the counting function of the roots of $f - a$ with multiplicities $> k$, where each a point is counted according to its multiplicity.

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \cup_{a\in S}\{z : f = a\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity, the set $\cup_{a\in S}\{z : f = a\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$, we say that f and g share the set S CM, and if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In order to validate our results, we require the following definitions and notations:

Definition 1.1 *[5] Let k be a non-negative integer, or infinitely. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.*

The definition implies that if f and g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write that f and g share (a, k) to mean that f and g share a value a with weight k. Clearly, if f and g share (a, k), then f and g share (a, p) for all integers p, $0 \leq p < k$. Also, we note that f and g share a value of a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞), respectively.

In 1997, C. C. Yang and X. H. Hua [11] investigated meromorphic functions sharing one value in response to Hayman's famous question [4].

Theorem A. Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, and let $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share 1 CM, then either $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = 1$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.

In 2002, Fang and Qiu [2] improved Theorem A by obtaining the following result:

Theorem B. Let $f(z)$ and $g(z)$ be two non-constant meromorphic (entire) functions, and let $n \geq 11$ and $(n \geq 6)$ be positive integers. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share z CM, then either $f(z) = c_1e^{cz^2}, g(z) = c_2e^{-cz^2}$, where c_1, c_2 and c are three constants satisfying $4(c_1c_2)^{n+1}c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} = 1$.

Wang and Gao [8] evolved the above conclusions for transcendental meromorphic functions with a small functions. They proved the following:

Theorem C. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let $a(z)(\not\equiv 0)$ be a common small function with respect to them and $n \geq 11$ be a positive integer. If $f^n(z)f'(z)$ and $g^n(z)g'(z)$ share $a(z)$ CM, then either $f^n(z)f'(z)g^n(z)g'(z) \equiv a^2(z)$ or $f(z) \equiv tg(z)$ for a constant such that $t^{n+1} =1$.

In 2020, Harina P. Waghamore and Ramya Maligi [9] improved and expanded Theorems A, B, and C by establishing the following:
Theorem D. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k+8$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share $(1,2)$, $f(z)$ and $g(z)$ share (∞, ∞), then one of the following three cases holds:

1. $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$,
2. $f^n f^{(k)} = g^n g^{(k)}$, if $\frac{f}{g}$ is not a constant:
3. $f = c_3 e^{dz}$ and $g = c_4 e^{-dz}$, where c_3, c_4 and d are constants such that $(-1)^k (c_3 c_4)^{n+1} d^{2k} = 1$.

Theorem E. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k+9$ be a positive integer. If $f^n f^{(k)}$ and $g^n g^{(k)}$ share $(1,2)$, $f(z)$ and $g(z)$ share $(\infty, 0)$, then the conclusion of Theorem D holds.
Theorem F. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k+8$ be a positive integer. If $f(z)$ and $g(z)$ share (∞, ∞) and $E_{3)}(1, f^n f^{(k)}) = E_{3)}(1, g^n f^{(k)})$, then the conclusion of Theorem D holds.
Theorem G. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions and let $n \geq k+9$ be a positive integer. If $f(z)$ and $g(z)$ share $(\infty, 0)$ and $E_{3)}(1, f^n f^{(k)}) = E_{3)}(1, g^n g^{(k)})$, then the conclusion of Theorem D holds.

Remark 1.2 *For* $S_1 = \{1, w, ..., w^{n-1}\}$ *and* $S_2 = \{\infty\}$, *the authors also proved the following equivalent forms:*

Theorem H. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that
$$E_{f(z)}(S_1, 2) = E_{g(z)}(S_1, 2),\ E_{f(z)}(S_2, \infty) = E_{g(z)}(S_2, \infty).$$
If $n \geq k+8$, then the conclusion of Theorem D holds.
Theorem I. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that
$$E_{f(z)}(S_1, 2) = E_{g(z)}(S_1, 2),\ E_{f(z)}(S_2, 0) = E_{g(z)}(S_2, 0).$$
If $n \geq k+9$, then the conclusion of Theorem D holds.
Theorem J. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that
$$E_{3)}(S_1, f(z)) = E_{3)}(S_1, g(z)),\ E_{f(z)}(S_2, \infty) = E_{g(z)}(S_2, \infty).$$
If $n \geq k+8$, then the conclusion of Theorem D holds.
Theorem K. Let S_1, S_2 be given as above. Suppose $f(z)$ and $g(z)$ be two transcendental meromorphic functions such that
$$E_{3)}(S_1, f(z)) = E_{3)}(S_1, g(z)),\ E_{f(z)}(S_2, 0) = E_{g(z)}(S_2, 0).$$
If $n \geq k+9$, then the conclusion of Theorem D holds.

In the same paper, Harina P. Waghamore and Ramya Maligi [9] posed the following open problems.

Open problem. Can the lower bound of n be further reduced in Theorem D-K to get the same conclusion?

We are now recalling the definition that inspired us to compose this article.

Definition 1.3 *[6] Let $n_{0j}, n_{1j}, \cdots, n_{kj}$ be non-negative integers. The expression,*

$$M_j[f] = (f)^{n_{0j}}(f^{(1)})^{n_{1j}} \cdots (f^{(k)})^{n_{kj}},$$

is called a differential monomial generated by f of degree $\gamma_{M_j} = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$. Then the expression

$$P[f] = \sum_{j=1}^{l} a_j M_j[f], \tag{1.1}$$

where $T(r, a_j) = S(r, f)$ for $j = 1, 2, ..., l$ is called the differential polynomial generated by f of upper degree $\gamma_p = max_{1\leq j\leq l}\{\gamma_{M_j}\}$, lower degree $\underline{\gamma}_p = min_{1\leq j\leq l}\{\gamma_{M_j}\}$, and weight $\Gamma_P = max_{1\leq j\leq l}\{\Gamma_{M_j}\}$ and the order k (where k is the highest order of the derivative of f in $P[f]$).

Let σ denote $max_{1\leq j\leq l}\{\Gamma_{M_j} - \gamma_{M_j}\}$, i.e.,

$$\begin{aligned}\sigma &= max_{1\leq j\leq l}\sum_{i=0}^{k}[(i+1)-1]n_{ij} \\ &= max_{1\leq j\leq l}(n_{1j} + 2n_{2j} + \ldots + kn_{kj}).\end{aligned}$$

After studying certain articles related to the above definition, it is typical to ask the following question:

Question. What will be the conclusion if $f^n f^{(k)}$ is replaced by $f^n(f-1)^m P[f] (m \geq 2)$(resp. $g^n(g-1)^m P[g]$) in Theorem D-K?

Using the concept of multiplicity, we intend to determine the possible solutions to the above questions. We demonstrated the following results, which are the main results of this paper:

Theorem 1.4 *Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with zeros and poles of multiplicities at least s, where s is a positive integer. Let $(n - m - 3\gamma_p)\, s > 3\sigma + 6$ be a positive integer. If $f^n(z)(f-1)^m(z)P[f(z)]$ and $g^n(z)(g-1)^m(z)P[g(z)]$ share $(1, 2)$, $f(z)$ and $g(z)$ share (∞, ∞), then one of the following conclusion holds.*

1. *$f = tg$, for a constant t such that $t^d = 1$, where $d = gcd\{n+1, n+2, \ldots, n+m+1-i, \ldots, n+m+1\}$.*

2. *f and g satisifying the algebraic equation $R(f, g) = 0$, where*

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m P[\omega_1] - \omega_2^n(\omega_2 - 1)^m P[\omega_2].$$

3. $(f^n(f-1)^m P[f]) \, . \, (g^n(g-1)^m P[g]) \equiv 1.$

Theorem 1.5 *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer. Let* $(n-m-3\gamma_p)\, s > 3\sigma + 7$ *be a positive integer. If* $f^n(z)(f-1)^m(z)P[f(z)]$ *and* $g^n(z)(g-1)^m(z)P[g(z)]$ *share* $(1,2)$*,* $f(z)$ *and* $g(z)$ *share* $(\infty, 0)$*, then the conclusion of Theorem 1.4 holds.*

Theorem 1.6 *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer. Let* $(n-m-3\gamma_p)\, s > 3\sigma + 6$ *be a positive integer. If* $f(z)$ *and* $g(z)$ *share* (∞, ∞) *and* $E_{3)}(1, f^n(z)(f-1)^m(z)P[f(z)]) = E_{3)}(1, g^n(z)(g-1)^m(z)P[g(z)])$*, then the conclusion of Theorem 1.4 holds.*

Theorem 1.7 *Let* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer. Let* $(n-m-3\gamma_p)\, s > 3\sigma + 7$ *be a positive integer. If* $f(z)$ *and* $g(z)$ *share* $(\infty, 0)$ *and* $E_{3)}(1, f^n(z)(f-1)^m(z)P[f(z)]) = E_{3)}(1, g^n(z)(g-1)^m(z)P[g(z)])$*, then the conclusion of Theorem 1.4 holds.*

From Remark 1.2, we can get the following equivalent forms.

Theorem 1.8 *Let* S_1*,* S_2 *be given in Remark 1.2. Suppose* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer such that*

$$E_{f(z)}(S_1, 2) = E_{g(z)}(S_1, 2),\ E_{f(z)}(S_2, \infty) = E_{g(z)}(S_2, \infty).$$

If $(n-m-3\gamma_p)\, s > 3\sigma + 6$*, then the conclusion of Theorem 1.4 holds.*

Theorem 1.9 *Let* S_1*,* S_2 *be given in Remark 1.2. Suppose* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer such that*

$$E_{f(z)}(S_1, 2) = E_{g(z)}(S_1, 2),\ E_{f(z)}(S_2, 0) = E_{g(z)}(S_2, 0).$$

If $(n-m-3\gamma_p)\, s > 3\sigma + 7$*, then the conclusion of Theorem 1.4 holds.*

Theorem 1.10 *Let* S_1*,* S_2 *be given in Remark 1.2. Suppose* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer such that*

$$E_{3)}(S_1, f(z)) = E_{3)}(S_1, g(z)),\ E_{f(z)}(S_2, \infty) = E_{g(z)}(S_2, \infty).$$

If $(n-m-3\gamma_p)\, s > 3\sigma + 6$*, then the conclusion of Theorem 1.4 holds.*

Theorem 1.11 *Let* S_1*,* S_2 *be given in Remark 1.2. Suppose* $f(z)$ *and* $g(z)$ *be two transcendental meromorphic functions with zeros and poles of multiplicities at least* s*, where* s *is a positive integer such that*

$$E_{3)}(S_1, f(z)) = E_{3)}(S_1, g(z)),\ E_{f(z)}(S_2, 0) = E_{g(z)}(S_2, 0).$$

If $(n-m-3\gamma_p)\, s > 3\sigma + 7$*, then the conclusion of Theorem 1.4 holds.*

Remark 1.12 *If suppose* $P[f] = f^{(k)}$, *then we get,*

$$(\gamma_p = \gamma_{M_1} = 1), (\Gamma_p = \Gamma_{M_1} = (k+1)), (\sigma = \Gamma_{M_1} - \gamma_{M_1} = k)$$

and in Theorems 1.4, 1.6, 1.8, and 1.10, giving specific values for s, m, *and* k *in the condition* $(n - m - 3\gamma_p)\, s > 3\sigma + 6$, *we get the following interesting cases:*

1. *If* $s = 1$ *and* $m = 0$, *then the results extend Theorems D, F, H, and J.*
2. *If* $s \geq 1$, $m = 0$, *and* $k = 1$ *in Theorems 1.4-1.11, generalize and improve Theorems A-K.*

Remark 1.13 *If suppose* $P[f] = f^{(k)}$, *then we get,*

$$(\gamma_p = \gamma_{M_1} = 1), (\Gamma_p = \Gamma_{M_1} = (k+1)), (\sigma = \Gamma_{M_1} - \gamma_{M_1} = k)$$

and in Theorems 1.5, 1.7, 1.9, and 1.11, giving specific values for s, m, *and* k *in the condition* $(n - m - 3\gamma_p)\, s > 3\sigma + 7$, *we get following interesting cases.*

1. *If* $s = 1$ *and* $m = 0$, *then the results extend Theorems E, G, I, and K.*
2. *If* $s \geq 1$, $m = 0$ *and* $k = 1$ *in Theorems 1.4-1.11, generalize and improve Theorems A-K.*

2 Some Preliminary Lemmas

In order to prove our results, we need the following lemmas.

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We define the function H as,

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Lemma 2.1 *[10] Let* f *be a non-constant meromorphic function and let* $a_1, a_2, a_3, ..., a_n$ *be finite complex numbers,* $a_n \neq 0$. *Then* $T(r, a_n f^n + ... + a_2 f^2 + a_1 f + a_0) = nT(r, f) + S(r, f)$.

Lemma 2.2 *[12] Let* $f(z)$ *be a non-constant meromorphic function and* $P[f]$ *be a differential polynomial if* f. *Then*

$$m\left(r, \frac{P[f]}{f^{\gamma_p}}\right) \leq \left(\gamma_p - \underline{\gamma}_p\right) m\Big(r, \frac{1}{f}\Big) + S(r, f),$$

$$m\Big(r, \frac{P[f]}{f^{\underline{\gamma}_p}}\Big) \leq \left(\gamma_p - \underline{\gamma}_p\right) m\Big(r, f\Big) + S(r, f),$$

$$N\left(r, \frac{P[f]}{f^{\gamma_p}}\right) \leq \left(\gamma_p - \underline{\gamma}_p\right) N\Big(r, \frac{1}{f}\Big) + \sigma\left[\overline{N}(r, f) + \overline{N}(r, \frac{1}{f})\right] + S(r, f),$$

$$N(r, P[f]) \leq \gamma_p N(r, f) + \sigma \overline{N}(r, f) + S(r, f),$$

$$T(r, P[f]) \leq \gamma_p T(r, f) + \sigma \overline{N}(r, f) + S(r, f),$$

where, $\sigma = \max\{n_{1j} + 2n_{2j} + 3n_{3j} + \cdots + kn_{kj}; 1 \le j \le l\}$.

Lemma 2.3 *[1] Let F, G be two non-constant meromorphic functions. If F, G share $(1,2)$ and (∞, k), where $0 \le k \le \infty$. If $H \not\equiv 0$, then*

$$T(r,F) \le N_2\Big(r, \frac{1}{F}\Big) + N_2\Big(r, \frac{1}{G}\Big) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) \\ + S(r,F) + S(r,G),$$

where $\bar{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those a-points of F whose multiplicities differ from the multiplicities of the corresponding a-points of G.

Lemma 2.4 *[1] Let F, G be two non-constant meromorphic functions. If F, G share (∞, k) and $E_{3)}(1,F) = E_{3)}(1,G)$, where $0 \le k \le \infty$. If $H \not\equiv 0$, then*

$$T(r,F) + T(r,G) \le 2N_2\Big(r, \frac{1}{F}\Big) + 2N_2\Big(r, \frac{1}{G}\Big) + 2\overline{N}(r,F) + 2\overline{N}(r,G) \\ + 2\overline{N_*}(r,\infty;F,G) + S(r,F) + S(r,G).$$

where $\bar{N}_*(r,\infty;F,G)$ denotes the reduced counting function of those a-points of F whose multiplicities differ from the multiplicities of the corresponding a-points of G.

3 Proof of the Main Results

Proof of Theorem 1.4. Let $F = f^n(f-1)^m P[f]$ and $G = g^n(g-1)^m P[g]$. Then F and G share $(1,2)$ and (∞,∞). Let H be defined as above. Suppose that $H \not\equiv 0$. From Lemma 2.3, we have

$$T(r,F) \le N_2\Big(r, \frac{1}{F}\Big) + N_2\Big(r, \frac{1}{G}\Big) + \overline{N}(r,F) + \overline{N}(r,G) + \overline{N_*}(r,\infty;F,G) \\ + S(r,F) + S(r,G). \tag{3.1}$$

It obvious that

$$\overline{N_*}(r,\infty;F,G) = 0$$

We deduce from Lemmas 2.1, 2.2, and (3.1) that

$$T(r,F) \le 2\overline{N}\Big(r, \frac{1}{f}\Big) + mN\Big(r, \frac{1}{f}\Big) + N\Big(r, \frac{1}{P[f]}\Big) + \overline{N}(r,f) + 2\overline{N}\Big(r, \frac{1}{g}\Big) \\ + mN\Big(r, \frac{1}{g}\Big) + N\Big(r, \frac{1}{P[g]}\Big) + \overline{N}(r,g) + S(r,f) + S(r,g) \\ \le \Big(\gamma_p + \frac{\sigma}{s} + m + \frac{3}{s}\Big)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g). \tag{3.2}$$

Obviously,

$$T(r,F) = T\Big(r, f^n(f-1)^m P[f]\Big) + S(r,f).$$

And

$$\begin{aligned}(n+m)T(r,f) &= T(r, f^n(f-1)^m) + S(r,f)\\ &= T\left(r, \frac{f^n(f-1)^m P[f]}{P[f]}\right) + S(r,f)\\ &\le T(r,F) + T(r,P[f]) + S(r,f)\\ &\le T(r,F) + \left(\gamma_p + \frac{\sigma}{s}\right) T(r,f) + S(r,f),\end{aligned}$$

which implies that

$$\left(n+m-\gamma_p-\frac{\sigma}{s}\right) T(r,f) \le T(r,F) + S(r,f). \tag{3.3}$$

Similarly, we can write for g also.

Now, from (3.2) and (3.3), we get

$$\begin{aligned}\left(n+m-\gamma_p-\frac{\sigma}{s}\right) T(r,f) \le \left(\gamma_p + \frac{\sigma}{s} + m + \frac{3}{s}\right) \{T(r,f)+T(r,g)\}\\ + S(r,f) + S(r,g).\end{aligned} \tag{3.4}$$

Similarly,

$$\begin{aligned}\left(n+m-\gamma_p-\frac{\sigma}{s}\right) T(r,g) \le \left(\gamma_p + \frac{\sigma}{s} + m + \frac{3}{s}\right) \{T(r,f)+T(r,g)\}\\ + S(r,f) + S(r,g).\end{aligned} \tag{3.5}$$

Adding (3.4) and (3.5), we get

$$\left(n-m-\frac{6}{s}-3\gamma_p-\frac{3\sigma}{s}\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which contradicts the condition $(n-m-3\gamma_p)\,s > 3\sigma+6$. Therefore $H \equiv 0$. By integration, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \tag{3.6}$$

where $A \neq 0$ and B are constants. From (3.6), we have

$$G = \frac{(B-A)F + (A-B-1)}{BF-B-1}. \tag{3.7}$$

Let us discuss the following three cases.
Case 1. Suppose that $B \neq 0, -1$. From (3.7), we have

$$\overline{N}\left(r, \frac{1}{F - \frac{B+1}{B}}\right) = \overline{N}(r, G). \tag{3.8}$$

From Nevanlinna's Fundamental Theorem -II and Lemma 2.2, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}\Big(r,F\Big) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}\left(r,\frac{1}{F-\frac{B+1}{B}}\right) + S(r,F) \\ &= \overline{N}(r,F) + \overline{N}\Big(r,\frac{1}{F}\Big) + \overline{N}(r,G) + S(r,F) + S(r,G) \\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\Big(r,\frac{1}{P[f]}\Big) + \overline{N}(r,f) + \overline{N}(r,g) + S(r,f) + S(r,g) \\ &\leq 2\overline{N}\left(r,\frac{1}{f}\right) + N\Big(r,\frac{1}{P[f]}\Big) + \overline{N}(r,f) + \overline{N}(r,g) + S(r,f) + S(r,g) \\ &\leq \left(\gamma_p + \frac{3+\sigma}{s}\right) T(r,f) + \frac{1}{s}T(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.9}$$

Then substituting (3.9) in (3.3), we get

$$\begin{aligned} \Big(n+m-\gamma_p-\frac{\sigma}{s}\Big) T(r,f) \leq \left(\gamma_p + \frac{3+\sigma}{s}\right) T(r,f) + \frac{1}{s}T(r,g) \\ + S(r,f) + S(r,g). \end{aligned} \tag{3.10}$$

Similarly, we can get

$$\begin{aligned} \Big(n+m-\gamma_p-\frac{\sigma}{s}\Big) T(r,g) \leq \left(\gamma_p + \frac{3+\sigma}{s}\right) T(r,g) + \frac{1}{s}T(r,f) \\ + S(r,f) + S(r,g). \end{aligned} \tag{3.11}$$

Adding (3.10) and (3.11), we get

$$\left(n+m-2\gamma_p-\frac{4-2\sigma}{s}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g),$$

which contradicts the condition $(n-m-3\gamma_p)\,s > 3\sigma + 6$.
Case 2. Suppose that $B = 0$. From (3.8), we have

$$G = AF - (A-1). \tag{3.12}$$

If $A \neq 1$, from (3.12), we obtain

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\Big(r, \frac{1}{G}\Big). \tag{3.13}$$

By proceeding as in the proof of Case 1, we obtain a contradiction. Thus $A = 1$. From (3.13) we have $F = G$, that is

$$f^n(z)(f-1)^m(z)P[f] = g^n(z)(g-1)^m(z)P[g]. \tag{3.14}$$

Let $h = \frac{f}{g}$. If h is a constant, then from (3.14) we write,

$$f^n P[f] \sum_{i=0}^{m} (-1)^i \binom{m}{m-i} f^{m-i} = g^n P[g] \sum_{i=0}^{m} (-1)^i \binom{m}{m-i} g^{m-i}. \tag{3.15}$$

Now, substituting $f = gh$ in (3.15), we get

$$\sum_{i=0}^{m} (-1)^i \binom{m}{m-i} g^{m-i} \left[h^{n+m-i+1} - 1\right] \equiv 0,$$

which implies that $h^d = 1$, where $d = gcd\{n+1, n+2, \ldots, n+m+1-i, \ldots, n+m+1\}$. Thus $f = tg$ for a constant t such that $t^d = 1$, where $d = gcd\{n+1, n+2, \ldots, n+m+1-i, \ldots, n+m+1\}$.

suppose that h is not a constant then from (3.14), we can say that f and g satisfies the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^n(w_1 - 1)^m P[w_1] - w_2^n(w_2 - 1)^m P[w_2].$$

Csae 3. Suppose that $B = -1$. From (3.8), we have

$$G = \frac{(A+1)F - A}{F}. \tag{3.16}$$

If $A \neq -1$, we obtain from (3.16) that

$$\overline{N}\left(r, \frac{1}{F - \frac{A-1}{A}}\right) = \overline{N}\left(r, \frac{1}{G}\right).$$

By proceeding as in the proof of Case 1, we obtain a contradiction. Hence $A = -1$. From (3.16), we get $FG = 1$ that is

$$\left(f^n(f-1)^m P[f]\right) . \left(g^n(g-1)^m P[g]\right) = 1.$$

which is one of the conclusion.

Proof of Theorem 1.5 Let $F = f^n(f-1)^m P[f]$ and $G = g^n(g-1)^m P[g]$. Then F and G share $(1, 2)$ and $(\infty, 0)$. Let H be defined as above. Suppose that $H \not\equiv 0$. From Lemma 2.3, we have

$$\begin{aligned} T(r, F) \leq\ & N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + \overline{N}(r, G) + \bar{N}_*(r, \infty; F, G) \\ & + S(r, F) + S(r, G). \end{aligned} \tag{3.17}$$

It obvious that

$$\overline{N_*}(r,\infty;F,G) \le \overline{N}(r,f). \tag{3.18}$$

Combining (3.2), (3.3), (3.17), and (3.18), we deduce

$$\begin{aligned}&\Big(n+m-\gamma_p-\frac{\sigma}{s}\Big)\,T(r,f)\\ &\le \left(\gamma_p+m+\frac{4+\sigma}{s}\right)T(r,f)+\left(\gamma_p+m+\frac{3+\sigma}{s}\right)T(r,g)+S(r,f)+S(r,g).\end{aligned} \tag{3.19}$$

Similarly, we can get

$$\begin{aligned}&\Big(n+m-\gamma_p-\frac{\sigma}{s}\Big)\,T(r,g)\\ &\le \left(\gamma_p+m+\frac{4+\sigma}{s}\right)T(r,g)+\left(\gamma_p+m+\frac{3+\sigma}{s}\right)T(r,f)+S(r,f)+S(r,g).\end{aligned} \tag{3.20}$$

Adding (3.19) and (3.20), we get

$$\left(n-m-3\gamma_p-\frac{3\sigma-7}{s}\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which contradicts the condition $(n-m-3\gamma_p)\,s > 3\sigma+7$.

Similar to the cases discussed in Theorem 1.4, we see that Theorem 1.5 holds.

Proof of Theorem 1.6 Let $F = f^n(f-1)^m P[f]$ and $G = g^n(g-1)^m P[g]$. Then F and G share $E_{3)}(1,F) = E_{3)}(1,G)$ and (∞,∞). So $\overline{N_*}(r,\infty;F,G) = 0$. Let H be defined as above. Suppose that $H \not\equiv 0$. From Lemma 2.4, we have

$$\begin{aligned}T(r,F)+T(r,G) \le\ & 2N_2\Big(r,\frac{1}{F}\Big)+2N_2\Big(r,\frac{1}{G}\Big)+2\overline{N}(r,F)+2\overline{N}(r,G)\\ &+2\overline{N_*}(r,\infty;F,G)+S(r,F)+S(r,G).\end{aligned} \tag{3.21}$$

We deduce from (3.21) that

$$\begin{aligned}T(r,F)+T(r,G) \le\ & 4\overline{N}\Big(r,\frac{1}{f}\Big)+2mN\Big(r,\frac{1}{f}\Big)+2N\Big(r,\frac{1}{P[f]}\Big)+2\overline{N}(r,f)\\ &+4\overline{N}\Big(r,\frac{1}{g}\Big)+2mN\Big(r,\frac{1}{g}\Big)+2N\Big(r,\frac{1}{P[g]}\Big)+2\overline{N}(r,g)\\ &+S(r,f)+S(r,g).\end{aligned} \tag{3.22}$$

Using (3.3) and (3.22), we deduce

$$\left(n-m-3\gamma_p-\frac{3\sigma-6}{s}\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which contradicts the condition $(n-m-3\gamma_p)\,s > 3\sigma+6$. Therefore $H \equiv 0$.

Similar to the cases discussed in Theorem 1.4, we see that Theorem 1.6 holds.

Proof of Theorem 1.7. Let $F = f^n(f-1)^m P[f]$ and $G = g^n(g-1)^m P[g]$. Then F and G share $E_{3)}(1,F) = E_{3)}(1,G)$ and $(\infty, 0)$. So $\bar{N}_*(r,\infty;F,G) \le \overline{N}(r,\infty;F) = \overline{N}(r,\infty;G)$. Let H be defined as above. Suppose that $H \not\equiv 0$. From Lemma 2.4, we have

$$T(r,F)+T(r,G) \le 2N_2\Big(r,\frac{1}{F}\Big) + 2N_2\Big(r,\frac{1}{G}\Big) + 3\overline{N}(r,F) + 3\overline{N}(r,G) + S(r,F) + S(r,G). \tag{3.23}$$

We deduce from (3.23), that

$$\begin{aligned} T(r,F)+T(r,G) \le{}& 4\overline{N}\Big(r,\frac{1}{f}\Big) + 2mN\Big(r,\frac{1}{f}\Big) + 2N\Big(r,\frac{1}{P[f]}\Big) \\ &+ 3\overline{N}(r,f) + 4\overline{N}\Big(r,\frac{1}{g}\Big) + 2mN\Big(r,\frac{1}{g}\Big) + 2N\Big(r,\frac{1}{P[g]}\Big) \\ &+ 3\overline{N}(r,g) + S(r,f) + S(r,g). \end{aligned} \tag{3.24}$$

Using (3.3) and (3.24), we deduce

$$\left(n-m-3\gamma_p - \frac{3\sigma-7}{s}\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g),$$

which contradicts the condition $(n-m-3\gamma_p)\,s > 3\sigma+7$. Therefore $H \equiv 0$. Similar to the cases discussed in Theorem 1.4, we see that Theorem 1.7 holds.

OPEN PROBLEM.

1. Can we expect the same conclusion by taking the difference-differential polynomial $P[f]$ as in [7] and a polynomial $P(f)$ of degree m in place of the differential polynomial and $(f-1)^m$, respectively?

2. Is there any alternative method that reduces the condition n in all the theorems?

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