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A note on how square root inequalities can be self-improving and self-expanding

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Abstract

This article is devoted to certain square root inequalities related to the reciprocal of the square root function. We show that some of them have the interesting property of being self-improving and self-extending. Known results on this topic are revisited and new ones are established. An open question is also raised, the relevance of which is supported by a graphical analysis.

Keywords: *square roots inequalities, convex inequalities, differentiation, integration.*
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1 Introduction

Inequalities involving square roots are fundamental to mathematical analysis. See [1], [5], [2], and [6]. In particular, they appear in various contexts such as numerical methods, probability theory and approximation theory. A famous chain of square root inequalities involving the reciprocal of the square root function, i.e., $1/\sqrt{x}$, is presented in the lemma below.

Lemma 1.1 *For any $x \geq 1$, we have*

$$2\left[\sqrt{x+1}-\sqrt{x}\right] \leq \frac{1}{\sqrt{x}} \leq 2\left[\sqrt{x}-\sqrt{x-1}\right].$$

This lemma can be found without proof in [7]. The inequalities presented are mainly of theoretical interest, but they also have practical applications. Indeed, thanks to an acceptable degree of precision, they can be used to estimate errors in numerical methods, to bound probabilities in statistical analysis, and to provide approximations in various branches of analysis.

In this article, we discuss some under-commented and new aspects of the inequalities in Lemma 1.1. More precisely, we have three objectives: First, we show that these inequalities have the ability to improve themselves thanks to well-chosen simple manipulations; second, they are not as rigid as they seem at first sight: some adaptable parameters can be introduced, relaxing the assumption "$x \geq 1$"; and third, there is room for improvement for the bounds of $1/\sqrt{x}$. A conjecture leading to an open question on this last aspect is formulated. It is supported by an extensive graphical analysis.

The next section, Section 2, focuses on results derived from Lemma 1.1 from an original point of view. Section 3 contains some new developments on the topic, including a conjecture. Section 4 contains some open problems on the topic. A conclusion is given in Section 5.

2 On Lemma 1.1

This section begins with an elegant proof of Lemma 1.1. It is followed by a refinement and extensions based only on this lemma.

2.1 An elegant proof

There are several ways to prove Lemma 1.1. An elegant, intuitive and "one-piece" proof is given by means of integrals. Using the fact that $1/\sqrt{t}$, $t \geq 1$, is decreasing, we get

$$\sqrt{x+1} - \sqrt{x} = \int_x^{x+1} \frac{1}{2\sqrt{t}} dt \leq \frac{1}{2\sqrt{x}} \int_x^{x+1} dt = \frac{1}{2\sqrt{x}}[(x+1) - x] = \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}}[x - (x-1)] = \frac{1}{2\sqrt{x}} \int_{x-1}^{x} dt \leq \int_{x-1}^{x} \frac{1}{2\sqrt{t}} dt = \sqrt{x} - \sqrt{x-1}.$$

If we multiply all the terms by 2, and keep the extreme terms and $1/\sqrt{x}$, we get

$$2\left[\sqrt{x+1} - \sqrt{x}\right] \leq \frac{1}{\sqrt{x}} \leq 2\left[\sqrt{x} - \sqrt{x-1}\right].$$

The desired result is achieved.

2.2 A refinement and an extension

The result below is an improvement of Lemma 1.1. It is a known result. For example, it appears in [4] and is simply called "square root inequalities". The interesting thing is that we prove it using only the inequalities in Lemma 1.1. In a sense, Lemma 1.1 refines itself.

Proposition 2.1 *The square root inequalities in Lemma 1.1 are self-improving; for any $x \geq 1$, we have*

$$2\left[\sqrt{x+1}-\sqrt{x}\right] \leq \frac{1}{\sqrt{x}} \leq \sqrt{x+1}-\sqrt{x-1} \leq 2\left[\sqrt{x}-\sqrt{x-1}\right].$$

Proof. Taking into account the inequalities in Lemma 1.1, it is sufficient to prove the two inequalities on the right side, i.e.,

$$\sqrt{x+1}-\sqrt{x-1} \leq 2\left[\sqrt{x}-\sqrt{x-1}\right] \tag{1}$$

and

$$\frac{1}{\sqrt{x}} \leq \sqrt{x+1}-\sqrt{x-1}. \tag{2}$$

From the extreme terms in the inequalities in Lemma 1.1, it immediately follows that

$$\sqrt{x+1}-\sqrt{x} \leq \sqrt{x}-\sqrt{x-1}, \tag{3}$$

which can be reformulated as

$$\sqrt{x+1}+\sqrt{x-1} \leq 2\sqrt{x}. \tag{4}$$

On the basis of these inequalities, let us consider Equations (1) and (2) in turn. By applying Equation (3), we have

$$\begin{aligned} 2\left[\sqrt{x}-\sqrt{x-1}\right] &= \sqrt{x}-\sqrt{x-1}+\sqrt{x}-\sqrt{x-1} \\ &\geq \sqrt{x}-\sqrt{x-1}+\sqrt{x+1}-\sqrt{x} = \sqrt{x+1}-\sqrt{x-1}. \end{aligned}$$

Equation (1) is proved.

On the other hand, with the help of Equation (4) and a suitable conjugate, we have

$$\begin{aligned} \sqrt{x+1}-\sqrt{x-1} &= \frac{\left[\sqrt{x+1}-\sqrt{x-1}\right]\left[\sqrt{x+1}+\sqrt{x-1}\right]}{\sqrt{x+1}+\sqrt{x-1}} \\ &= \frac{(x+1)-(x-1)}{\sqrt{x+1}+\sqrt{x-1}} = \frac{2}{\sqrt{x+1}+\sqrt{x-1}} \geq \frac{2}{2\sqrt{x}} = \frac{1}{\sqrt{x}}. \end{aligned}$$

Equation (2) is established. This ends the proof of the proposition. □

Thus, thanks to the introduction of the term $\sqrt{x+1}-\sqrt{x-1}$ into the inequalities of Lemma 1.1, we have shown that they have the ability to improve themselves; only the inequalities of Lemma 1.1 are used in the proof of Proposition 2.1 and $\sqrt{x+1}-\sqrt{x-1}$ does not involve $1/\sqrt{x}$. This is an under-commented aspect to the best of our knowledge. More in this direction will be done in Section 3.

Proposition 2.1 may seem restrictive because we need $x \geq 1$. In the proposition below, we see how the inequalities of this proposition can be self-expanding by using two adjustable parameters, α and β, and considering $\beta/\sqrt{x+\alpha}$ as the central term.

Proposition 2.2 *For any $\beta > 0$, $\alpha \in \mathbb{R}$, and $x \geq \beta - \alpha$, we have*

$$2\left[\sqrt{x+\alpha+\beta}-\sqrt{x+\alpha}\right] \leq \frac{\beta}{\sqrt{x+\alpha}} \leq \sqrt{x+\alpha+\beta}-\sqrt{x+\alpha-\beta}$$
$$\leq 2\left[\sqrt{x+\alpha}-\sqrt{x+\alpha-\beta}\right].$$

Proof. For any $y \geq \beta - \alpha$, let us set

$$x = \frac{y+\alpha}{\beta}.$$

Then we have $x \geq 1$ and Proposition 2.1 applied with this specific x gives

$$2\left[\sqrt{\frac{y+\alpha}{\beta}+1}-\sqrt{\frac{y+\alpha}{\beta}}\right] \leq \frac{1}{\sqrt{(y+\alpha)/\beta}} \leq \sqrt{\frac{y+\alpha}{\beta}+1}-\sqrt{\frac{y+\alpha}{\beta}-1}$$
$$\leq 2\left[\sqrt{\frac{y+\alpha}{\beta}}-\sqrt{\frac{y+\alpha}{\beta}-1}\right].$$

By arranging some ratio terms, we get

$$2\left[\sqrt{\frac{y+\alpha+\beta}{\beta}}-\sqrt{\frac{y+\alpha}{\beta}}\right] \leq \frac{\sqrt{\beta}}{\sqrt{y+\alpha}} \leq \sqrt{\frac{y+\alpha+\beta}{\beta}}-\sqrt{\frac{y+\alpha-\beta}{\beta}}$$
$$\leq 2\left[\sqrt{\frac{y+\alpha}{\beta}}-\sqrt{\frac{y+\alpha-\beta}{\beta}}\right].$$

Multiplication of all terms by $\sqrt{\beta}$ results in

$$2\left[\sqrt{y+\alpha+\beta}-\sqrt{y+\alpha}\right] \leq \frac{\beta}{\sqrt{y+\alpha}} \leq \sqrt{y+\alpha+\beta}-\sqrt{y+\alpha-\beta}$$
$$\leq 2\left[\sqrt{y+\alpha}-\sqrt{y+\alpha-\beta}\right].$$

Just replacing "y" by "x" for notation reasons, we end the proof. □

By taking $\alpha = 0$ and $\beta = 1$, Proposition 2.2 becomes Proposition 2.1. If we analyze his proof of Proposition 2.2, we see that Proposition 2.1 has extended itself with a little effort (and the same can be said for Lemma 1.1).

To conclude this section, we have shown that Lemma 1.1 can be self-improving and self-expanding, which remains a fascinating mathematical fact. The rest of the article is devoted to some new developments inspired by this lemma.

3 Some new developments

3.1 A new chain of inequalities

The proposition below presents a new chain of inequalities involving $1/\sqrt{x}$ and $2\left[\sqrt{x+1}-\sqrt{x}\right]$, and some intermediate square root and power terms.

Proposition 3.1 *For any $x \geq 1$, we have*

$$\begin{aligned}
&2\left[\sqrt{x+1}-\sqrt{x}\right] \leq \frac{2}{3}\left[(x+2)^{3/2}-x^{3/2}-(x+1)^{3/2}+(x-1)^{3/2}\right] \\
&\leq \frac{1}{2}\left[\sqrt{x+2}-\sqrt{x}+\sqrt{x+1}-\sqrt{x-1}\right] \leq \kappa\frac{1}{\sqrt{x}},
\end{aligned}$$

where

$$\kappa = \frac{1}{2}\left[1+\sqrt{2}\right] \approx 1.2071.$$

Proof. Let us start with the first inequality on the left side. It follows from Proposition 2.1 that, for any $t \geq 1$,

$$\frac{1}{\sqrt{t}} \leq \sqrt{t+1}-\sqrt{t-1}.$$

Hence, we get

$$\begin{aligned}
&2\left[\sqrt{x+1}-\sqrt{x}\right] = \int_x^{x+1}\frac{1}{\sqrt{t}}dt \leq \int_x^{x+1}\left[\sqrt{t+1}-\sqrt{t-1}\right]dt \\
&= \left[\frac{2}{3}(t+1)^{3/2}-\frac{2}{3}(t-1)^{3/2}\right]_{t=x}^{t=x+1} \\
&= \frac{2}{3}\left[(x+2)^{3/2}-x^{3/2}-(x+1)^{3/2}+(x-1)^{3/2}\right].
\end{aligned}$$

The desired inequality is demonstrated.

We now consider the second inequality on the left side. Let us set $f(t) = \sqrt{t+1} - \sqrt{t-1}$, $t \geq 1$. We then have

$$f''(t) = \frac{(t+1)^{3/2} - (t-1)^{3/2}}{4(t-1)^{3/2}(t+1)^{3/2}} \geq 0,$$

which means that $f(t)$ is convex. From the bound on the right side of the Hermite-Hadamard inequality applied to the function $f(t)$, $a = x$ and $b = x+1$, it follows that

$$\frac{2}{3}\left[(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2}\right] = \int_x^{x+1} \left[\sqrt{t+1} - \sqrt{t-1}\right] dt$$
$$= \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{1}{2}\left[f(b) + f(a)\right]$$
$$= \frac{1}{2}\left[f(x+1) + f(x)\right] = \frac{1}{2}\left[\sqrt{x+2} - \sqrt{x} + \sqrt{x+1} - \sqrt{x-1}\right].$$

See [3] for more details on the Hermite-Hadamard inequality.

For the last inequality to prove, by the use of appropriate conjugates, we have

$$\begin{aligned}
&\frac{1}{2}\left[\sqrt{x+2} - \sqrt{x} + \sqrt{x+1} - \sqrt{x-1}\right] \\
&= \frac{1}{2}\Bigg[\frac{\left[\sqrt{x+2} - \sqrt{x}\right]\left[\sqrt{x+2} + \sqrt{x}\right]}{\sqrt{x+2} + \sqrt{x}} \\
&+ \frac{\left[\sqrt{x+1} - \sqrt{x-1}\right]\left[\sqrt{x+1} + \sqrt{x-1}\right]}{\sqrt{x+1} + \sqrt{x-1}}\Bigg] \\
&= \frac{1}{2}\left[\frac{2}{\sqrt{x+2} + \sqrt{x}} + \frac{2}{\sqrt{x+1} + \sqrt{x-1}}\right] \\
&= \frac{1}{\sqrt{x+2} + \sqrt{x}} + \frac{1}{\sqrt{x+1} + \sqrt{x-1}}.
\end{aligned}$$

We clearly have

$$\sqrt{x+2} + \sqrt{x} \geq \sqrt{x} + \sqrt{x} = 2\sqrt{x}.$$

On the other hand, by using the famous square root inequality $\sqrt{u+v} \leq \sqrt{u} + \sqrt{v}$, for $u \geq 0$ and $v \geq 0$, we have

$$\sqrt{x+1} + \sqrt{x-1} \geq \sqrt{x+1+x-1} = \sqrt{2x} = \sqrt{2}\sqrt{x}.$$

Therefore, we get

$$\frac{1}{2}\left[\sqrt{x+2} - \sqrt{x} + \sqrt{x+1} - \sqrt{x-1}\right] \leq \frac{1}{2}\left[1 + \sqrt{2}\right]\frac{1}{\sqrt{x}} = \kappa\frac{1}{\sqrt{x}}.$$

The desired inequalities are demonstrated. The proof is complete. □

For any $x \geq x_* \approx 1.034$, numerical analysis shows that we can improve κ and replace it with $\kappa_* = 1$. The rigorous proof of this claim is not given here.

Proposition 3.1 is not a direct improvement of Proposition 2.1, but provides alternative inequalities related to $1/\sqrt{x}$ that can be used in several mathematical scenarios. To the best of our knowledge, these inequalities are new.

Some refinements and a conjecture are the subject of the next part.

3.2 Some refinements

The proposition below presents a new chain of inequalities involving power terms, and a related conjecture involving $1/\sqrt{x}$ and $\sqrt{x+1}-\sqrt{x-1}$, and some intermediate power terms.

Proposition 3.2 *The following inequality holds: For any $x \geq 1$, we have*

$$\frac{2}{3}\left[(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2}\right] \leq \frac{4}{3}\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right],$$

or, equivalently,

$$(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2} \leq 2\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right].$$

To complete this result, we formulate a conjecture below.
Conjecture: The above inequality is interesting because a graphical analysis of it shows that, for any $x \geq 1$, we have

$$\frac{2}{3}\left[(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2}\right] \leq \frac{1}{\sqrt{x}} \leq \frac{4}{3}\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right] \leq \sqrt{x+1} - \sqrt{x-1}, \tag{5}$$

and these inequalities are numerically sharp (a graphical proof is given).

Proof. An inequality in Proposition 2.1 states that, for any $t \geq 1$,

$$\sqrt{t+1} - \sqrt{t-1} \leq 2\left[\sqrt{t} - \sqrt{t-1}\right].$$

Therefore, we have

$$\begin{aligned}
&\frac{2}{3}\left[(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2}\right] = \int_x^{x+1} \left[\sqrt{t+1} - \sqrt{t-1}\right] dt \\
&\leq 2\int_x^{x+1} \left[\sqrt{t} - \sqrt{t-1}\right] dt = 2\left[\frac{2}{3}t^{3/2} - \frac{2}{3}(t-1)^{3/2}\right]_{t=x}^{t=x+1} \\
&= \frac{4}{3}\left[(x+1)^{3/2} - x^{3/2} - x^{3/2} + (x-1)^{3/2}\right] \\
&= \frac{4}{3}\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right].
\end{aligned}$$

The first inequality is proved.

Let us now illustrate the proposed conjecture with a graphical analysis. To do this, we set

$$p(x) = \frac{1}{\sqrt{x}} - \frac{2}{3}\left[(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2}\right],$$

$$q(x) = \frac{4}{3}\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right] - \frac{1}{\sqrt{x}}$$

and

$$r(x) = \sqrt{x+1} - \sqrt{x-1} - \frac{4}{3}\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right].$$

Figures 1, 2 and 3 show the curves of these three functions for different ranges of $x \geq 1$, respectively.

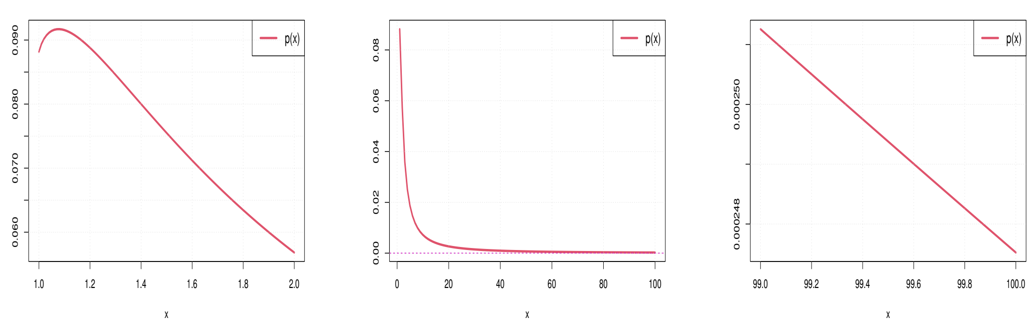

Figure 1: Curves of $p(x)$ for $x \in [1, 2)$ for a focus on the behavior around $x = 1$ (left), for $x \in [1, 100)$ for a global view (middle), and for $x \in [99, 100)$ for a focus on the behavior around $x = 100$, which is considered as a large value (right)

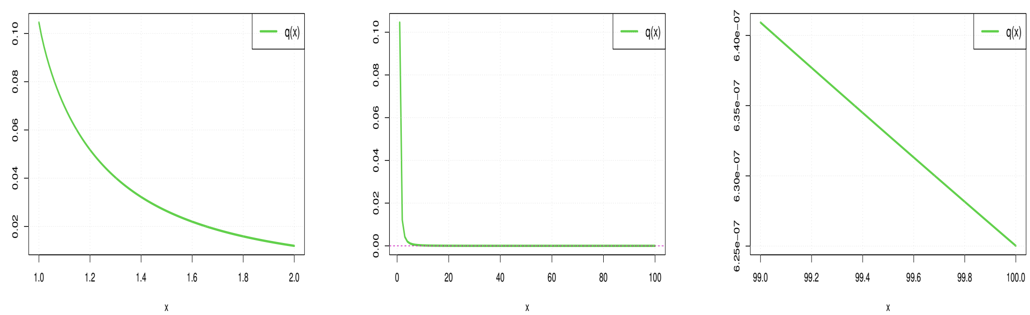

Figure 2: Curves of $q(x)$ for $x \in [1, 2)$ for a focus on the behavior around $x = 1$ (left), for $x \in [1, 100)$ for a global view (middle), and for $x \in [99, 100)$ for a focus on the behavior around $x = 100$ (right)

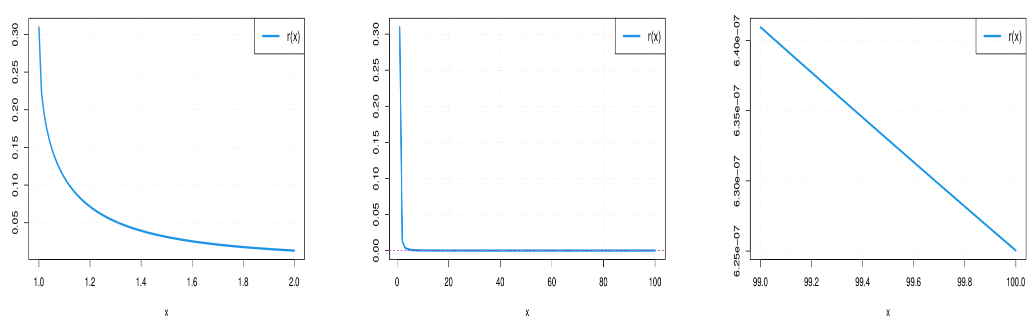

Figure 3: Curves of $r(x)$ for $x \in [1, 2)$ for a focus on the behavior around $x = 1$ (left), for $x \in [1, 100)$ for a global view (middle), and for $x \in [99, 100)$ for a focus on the behavior around $x = 100$ (right)

From these figures, it is clear that $p(x) \geq 0$, $q(x) \geq 0$ and $r(x) \geq 0$ for any $x \in [1, 100]$, and, of course, further zoomed graphical tests confirm this for any values of $x \geq 100$. This ends the graphical analysis. $\square$

If we analyze the proof of Proposition 3.2 and the graphical verification of the conjecture, this again supports the fact that the square root inequalities in Proposition 2.1 are self-improving. It also proves that there is room for mathematical improvement for square root inequalities centered on $1/\sqrt{x}$.

4 Open problems

An open problem is to find $a_1 \in \mathbb{R}$, $b_1 \in \mathbb{R}$, $c_1 \in \mathbb{R}$ and $d_1 \in \mathbb{R}$ such that the following inequality is sharp:

$$\frac{1}{\sqrt{x}} \leq a_1 \left[\sqrt{x+b_1} + c_1\sqrt{x+d_1}\right]$$

and $a_1 \left[\sqrt{1+b_1} + c_1\sqrt{1+d_1}\right] = 1$ to correspond to the value of $1/\sqrt{x}$ at $x = 1$. This simple constraint on the parameters is not satisfied by the inequalities we have presented.

Similarly, another open problem is to find $a_2 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, $c_2 \in \mathbb{R}$ and $d_2 \in \mathbb{R}$ such that the following inequality is sharp:

$$a_2 \left[\sqrt{x+b_2} + c_2\sqrt{x+d_2}\right] \leq \frac{1}{\sqrt{x}}$$

and $a_2 \left[\sqrt{1+b_2} + c_2\sqrt{1+d_2}\right] = 1$.

On the other hand, the inequalities in Equation (5) have been proved graphically, which leads to the following open question: Can we prove the inequalities in Equation (5) with a rigorous analytical proof that complements the graphical analysis?

5 Conclusion

In this article, we have "taken a fresh look" at known square root inequalities centered on $1/\sqrt{x}$, and innovated with some new ones. A focus is put on the interesting property of some such inequalities to be self-improving and self-expanding. Some of the results can be summarized in the following chain of inequalities: For any $x \geq 1$, we have

$$\begin{aligned}
&2\left[\sqrt{x+1} - \sqrt{x}\right] \leq \frac{2}{3}\left[(x+2)^{3/2} - x^{3/2} - (x+1)^{3/2} + (x-1)^{3/2}\right] \underbrace{\leq}_{\star} \frac{1}{\sqrt{x}} \\
&\leq \frac{4}{3}\left[(x+1)^{3/2} - 2x^{3/2} + (x-1)^{3/2}\right] \underbrace{\leq}_{\star} \sqrt{x+1} - \sqrt{x-1} \\
&\leq 2\left[\sqrt{x} - \sqrt{x-1}\right], \qquad (6)
\end{aligned}$$

where $\star$ refers to graphically proven inequalities. This chain of inequalities is illustrated in Figure 4.

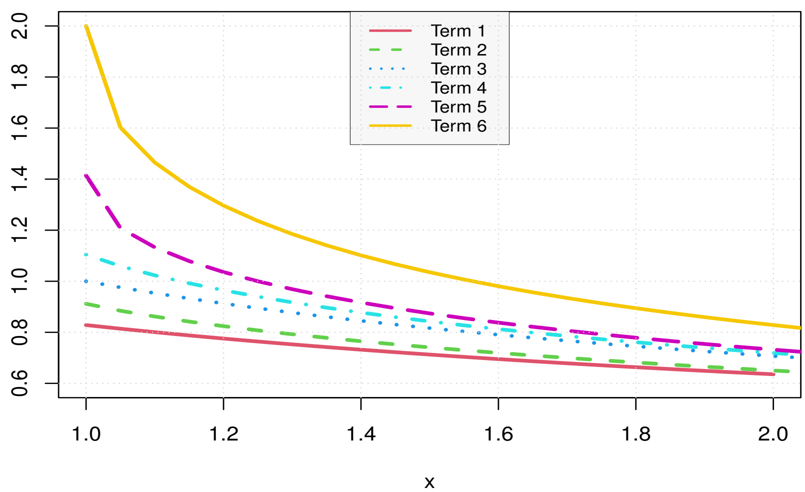

Figure 4: Curves of the functions in Equation (6), where Term 1 denotes the first function of the left side, Term 2 denotes the second function of the left side, and so on, for $x \in [1, 2]$

Our results raise mathematical questions about their rigorous analytical proofs. We believe that efforts can be made in this direction and further refinements can be demonstrated with some of the tools used in the article and others.

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