Int. J. Open Problems Compt. Math., Vol. 17, No. 2, June 2024
Print ISSN: 1998-6262, Online ISSN: 2079-0376

On the Partial Sharing Values with Linear $c-$Difference Operators

Harina P. Waghamore And Manjunath B. E.

Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bangalore -560 056, India
e-mail: harinapw@gmail.com, manjunath.bebub@gmail.com

Received 10 January 2024; Accepted 3 May 2024

Abstract

We have proven unique results concerning the partial sharing values between the functions $F_{a,b}$ and the linear $c-$difference operator $\mathcal{L}_n(f, \Delta_c)$. Specifically, we have $F_{a,b} = af + b$, where $a(\neq 0)$, $b \in \mathbb{C}$, and $\mathcal{L}_n(f, \Delta_c) = \mathcal{L}_c^n(f) - \left(\sum_{i=0}^{n} a_i\right) f(z)$, where a_i's are constants and c is a non-zero constant. As a consequence of this, we have shown that $F_{a,b}$ is identical to its linear $c-$difference operator $\mathcal{L}_n(f, \Delta_c)$.

Keywords: *Difference operator, uniqueness, Entire function, weighted sharing.*
2010 Mathematics Subject Classification: 30D35, 39A32.

1 Background Information & Main Result

Throughout this paper, unless explicitly stated otherwise, the term 'meromorphic' is used exclusively to refer to non-constant functions that are analytic (holomorphic) over the entire complex plane, with the exception of isolated poles. A meromorphic function $a(z)$ is considered to be a small function relative to another meromorphic function $f(z)$ if the Nevanlinna characteristic $T(r, a)$ satisfies the relation $T(r, a) = S(r, f)$ as r approaches infinity, where $S(r, f)$ represents a term of smaller magnitude than $T(r, f)$. This small function condition holds unconditionally if $f(z)$ has finite order. However, if $f(z)$

does not have finite order, then the condition $T(r, a) = S(r, f)$ may not hold for certain values of r lying within a set of finite linear measure.

Definition 1.1 *Let $f(z)$ and $g(z)$ be two meromorphic functions defined in the complex plane $\mathbb{C}$. We say that $f(z)$ and $g(z)$ share the complex value a counting multiplicities (CM) if the functions $f(z) - a$ and $g(z) - a$ have precisely the same zeros, with each zero having the same multiplicity for both functions.*

Definition 1.2 *Consider two non-constant meromorphic functions, $f(z)$ and $g(z)$, defined in the complex plane, and let a be a complex number or the point at infinity. Denote by $E(a, f)$ the set of all zeros of the function $f(z)-a$, where each zero is counted according to its multiplicity. That is, if a zero has multiplicity m, it is counted m times in the set $E(a, f)$.*
We say that $f(z)$ and $g(z)$ partially share the value a counting multiplicities (CM) if the set $E(a, f)$ is a subset of the set $E(a, g)$. In other words, every zero of $f(z) - a$, with its corresponding multiplicity, is also a zero of $g(z) - a$ with the same multiplicity.
It is important to note that the condition $E(a, f) = E(a, g)$ represents the case where $f(z)$ and $g(z)$ share the value a counting multiplicities (CM) in the traditional sense. Therefore, the concept of partially sharing a value CM is a more general condition that encompasses the traditional notion of sharing a value CM.

To begin, let us introduce the shift and difference operators as follows: For a meromorphic function f, we denote its shift by $I_c f(z) = f(z + c)$ and the corresponding difference operator by $\Delta_c f(z) = (I_c - 1)f(z) = f(z+c) - f(z)$. Furthermore, we recursively define $\Delta_c^s = \Delta_c^{s-1} = (\Delta_c^s)$, $\forall s \in \mathbb{N} - \{1\}$. To extend these definitions, we introduce the linear shift operator as:

$$\mathcal{L}_c^n(f) = a_n f(z + nc) + an - 1f(z + (n-1)c) + \cdots + a_0 f(z)$$

where $a_n \neq 0, a_{n-1}, \ldots, a_0 \in \mathbb{C}$, and $c \in \mathbb{C}^*$. Next, we define the linear c-difference operator as:

$$\begin{aligned} \mathcal{L}_n(f, \Delta_c) &= a_n \Delta_{nc} f(z) + a_{n-1} \Delta_{(n-1)c} f(z) + \cdots + a_1 \Delta_c f(z) + a_0 \Delta_0 f(z) \\ &= \mathcal{L}_c^n(f) - \left(\sum_{i=0}^{n} a_i \right) f(z). \end{aligned} \tag{1}$$

For the specific choice of constants $a_i = (-1)^{n-i}\binom{n}{i}$, where $0 \leq i \leq n$, in the expression $\mathcal{L}_n(f, \Delta_c)$, it is evident that $\mathcal{L}_n(f, \Delta_c) = \Delta_c^n f$.

Lastly, we define $F_{a,b}$ by the expression:

$$F_{a,b} = af + b, \quad \text{where} \quad a(\neq 0),\ b \in \mathbb{C}. \tag{2}$$

In recent years, many researchers (e.g., [7, 5]) focused on studying the value distribution of analogues associated with meromorphic functions. Consequently, this has motivated numerous researchers to investigate the uniqueness problem concerning meromorphic functions that share values or sets with their corresponding shift or difference operators.

In 2013, Chen and Yi investigated a meromorphic function that shares three distinct values alongside its first-order difference operator, leading to the following outcomes.

Theorem 1.3 *[2] Let $f(z)$ be a transcendental meromorphic function such that its order of growth $\rho(f)$ is not an integer or infinite, and let $\eta \in \mathbb{C}$ be a constant such that $f(z+\eta) \not\equiv f(z)$. If $\Delta f = f(z+\eta) - f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM, then $f(z+\eta) \equiv 2f(z)$.*

In 2016, *Lü* demonstrated that the aforementioned Theorem 1.3 remains valid for meromorphic functions of finite order. This signifies

Theorem 1.4 *[4] Let $f(z)$ be a transcendental meromorphic function of finite order, let $\Delta f = f(z+c) - f(z)(\not\equiv 0)$, where $c \neq 0$ is a finite number. If Δf and $f(z)$ share three distinct values e_1, e_2, ∞ CM, then $\Delta f \equiv f$.*

In this paper, we extend the aforementioned theorems to linear difference polynomials and establish the following results.

Theorem 1.5 *Let $f(z)$ be a transcendental meromorphic function of finite order and $F_{a,b}$ and $\mathcal{L}_n(f, \Delta_c)$ be defined as in (2) and (1) respectively. If $\mathcal{L}_n(f, \Delta_c)$ and $F_{a,b}$ share the value $1CM$ and satisify $E(0, F_{a,b}) \subset E(0, \mathcal{L}_n(f, \Delta_c))$ and $E(\infty, F_{a,b}) \supset E(\infty, \mathcal{L}_n(f, \Delta_c))$ then $F_{a,b} \equiv \mathcal{L}_n(f, \Delta_c)$.*

2 Supporting Results

To prove our main results, we require the following Lemmas.

Lemma 2.1 *[3] Let $g(z)$ be a meromorphic function of finite order ρ, and let $c \in \mathbb{C}$ be fixed . Then for each $\epsilon > 0$, we have*

$$m\left(r, \frac{g(z+c)}{g(z)}\right) + m\left(r, \frac{g(z)}{g(z+c)}\right) = O(r^{\rho-1+\epsilon}).$$

Lemma 2.2 *[6] If $\mathcal{R}(f)$ is rational in f and has small meromorphic coefficients then*

$$T(r, \mathcal{R}(f)) = deg_f(\mathcal{R})T(r, f) + S(r, f).$$

Lemma 2.3 *Let $f(z)$ be a meromorphic function of finite order and $\mathcal{L}_n(f, \Delta_c)$, $F_{a,b}$ are defined as in (1) and (2). Then we have* *i)* $m\left(r, \frac{\mathcal{L}_n(f,\Delta_c)}{F_{a,b}}\right) = S(r,f)$,

ii) $m\left(r, \frac{\mathcal{L}_n(f,\Delta_c)}{F_{a.b}-d}\right) = S(r,f)$, *for any complex constant d.*

Proof:

i) Using Lemma 2.1, we deduce that

$$\begin{aligned}
&m\left(r, \frac{\mathcal{L}_n(f,\Delta_c)}{f}\right)\\
&= m\left(r, \frac{b_n(z)f(z+nc)+b_{n-1}(z)f(z+(n-1)c)+\cdots a_0(z)f(z)-\sum_{i=0}^{n} b_i f(z)}{F_{a,b}}\right)\\
&\leq S(r,f)
\end{aligned}$$

ii) By the above result, we have

$$m\left(r, \frac{\mathcal{L}_n(f-d,\ \Delta_c)}{F_{a,b}-d}\right) = S(r,f).$$

Since $\sum_{i=0}^{n} b_i = 0$, we get $\mathcal{L}_n(d,\Delta_c) = 0$. Hence

$$\mathcal{L}_n(f-d,\ \Delta_c) = \mathcal{L}_n(f,\Delta_c) - \mathcal{L}_n(d,\ \Delta_c) = \mathcal{L}_n(f,\Delta_c).$$

By combining the above two equations, we obtain

$$m\left(r, \frac{\mathcal{L}_n(f,\Delta_c)}{F_{a,b}-d}\right) = m\left(r, \frac{\mathcal{L}_n(f-d,\ \Delta_c)}{F_{a,b}-d}\right) = S(r,f).$$

Lemma 2.4 *Let $c \in \mathbb{C}$ and $\mathcal{L}_n(f,\Delta_c)$, $F_{a,b}$ be given by* (1) *and* (2) *such that $\sum_{i=0}^{n} b_i = 0$. Let $q \geq 2$ and let $e_1(z),\ e_2(z)\cdots, e_q(z)$ be distinct meromorphic functions with period C such that $e_k \in S(f)$ for all $k = 1, 2, \cdots, q$. Then*

$$m(r, F_{a,b}) + \sum_{k=1}^{q} m\left(r, \frac{1}{F_{a,b}-e_k}\right) \leq 2T(r,F_{a,b}) - N_*(r,F_{a,b}) + S(r,F_{a,b}),$$

where $N_*(r,F_{a,b}) = 2N(r,F_{a,b}) + N\left(r, \frac{1}{\mathcal{L}_n(f,\Delta_c)}\right) - N\left(r, \mathcal{L}_n(f,\Delta_c)\right)$.

Proof: By denoting $\Phi\left(F_{a,b}\right)=\prod\limits_{k=1}^{q}\left(F_{a,b}-e_k\right)$, we have

$$\frac{1}{\Phi\left(F_{a,b}\right)}=\sum_{k=1}^{q}\frac{g_k}{F_{a,b}-e_k}, \tag{3}$$

where $g_k\in S(f)$, such that

$$\sum_{k=1}^{q} g_k\prod_{j\in\wedge\setminus\{k\}}\left(F_{a,b}-e_k\right)=1,\quad \wedge=\{1,2,\cdots,q\}$$

Since $\mathcal{L}_n(F_{a,b},\Delta_c)=a\mathcal{L}_n(f,\Delta_c)$, in view of Lemma 2.1, a simple computation shows that

$$\begin{aligned}
m\left(r,\frac{1}{\Phi\left(F_{a,b}\right)}\right)&\le\sum_{k=1}^{q}m\left(r,\frac{g_k}{F_{a,b}-e_k}\right)+S(r,f)\\
&\le\sum_{k=1}^{q}m\left(r,\frac{1}{F_{a,b}-e_k}\right)+S(r,f).
\end{aligned}$$

Hence, by (ii) of Lemma 2.3, we obtain

$$\begin{aligned}
m\left(r,\frac{\mathcal{L}_n(f,\Delta_c)}{\Phi\left(F_{a,b}\right)}\right)&\le\sum_{k=1}^{q}m\left(r,\frac{\mathcal{L}_n(f,\Delta_c)}{F_{a,b}-e_k}\right)+S(r,F_{a,b})\\
&\le\sum_{k=1}^{q}m\left(r,\frac{\mathcal{L}_n(f,\Delta_c)}{a(F_{a,b}-e_k)}\right)+S(r,F_{a,b})\\
&\le S\left(r,F_{a,b}\right).
\end{aligned} \tag{4}$$

Therefore, it is easy to see that

$$\begin{aligned}
m\left(r,\frac{1}{\Phi\left(F_{a,b}\right)}\right)&=m\left(r,\frac{\mathcal{L}_n(f,\Delta_c)}{\Phi\left(F_{a,b}\right)}\frac{1}{\mathcal{L}_n(f,\Delta_c)}\right)\\
&\le m\left(r,\frac{\mathcal{L}_n(f,\Delta_c)}{\Phi\left(F_{a,b}\right)}\right)+m\left(r,\frac{1}{\mathcal{L}_n(f,\Delta_c)}\right)+S(r,F_{a,b})\\
&\le m\left(r,\frac{1}{\mathcal{L}_n(f,\Delta_c)}\right)+S(r,F_{a,b}).
\end{aligned} \tag{5}$$

By the first main theorem, using (5) and (3) in view of Lemma 2.2 an easy computation shows that

$$\begin{aligned}
T\left(r, \mathcal{L}_n(f, \Delta_c)\right) &= m\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + O(1)\\
&\geq m\left(r, \frac{1}{\Phi\left(F_{a,b}\right)}\right) + N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + S(r, F_{a,b})\\
&\geq T\left(r, \frac{1}{\Phi\left(F_{a,b}\right)}\right) - N\left(r, \frac{1}{\Phi\left(F_{a,b}\right)}\right) + N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right)\\
&+ S(r, F_{a,b})\\
&\geq qT(r, f) - \sum_{k=1}^{q} N\left(r, \frac{1}{F_{a,b} - e_k}\right) + N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right)\\
&+ S(r, F_{a,b})\\
&\geq \sum_{k=1}^{q} m\left(r, \frac{1}{F_{a,b} - e_k}\right) + N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + S(r, F_{a,b})
\end{aligned}$$

i.e.,

$$\sum_{k=1}^{q} m\left(r, \frac{1}{F_{a,b} - e_k}\right) \leq T\left(r, \mathcal{L}_n(f, \Delta_c)\right) - N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + S(r, F_{a,b}).$$

Therefore, we obtain

$$\begin{aligned}
m\left(r, F_{a,b}\right) + \sum_{k=1}^{q} m\left(r, \frac{1}{F_{a,b} - e_k}\right) &\leq T(r, F_{a,b}) - N\left(r, F_{a,b}\right) + m(r, \mathcal{L}_n(f, \Delta_c))\\
&\quad + N(r, \mathcal{L}_n(f, \Delta_c)) - N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + S(r, F_{a,b})\\
&\leq 2T(r, F_{a,b}) - 2N(r, F_{a,b}) + N(r, \mathcal{L}_n(f, \Delta_c))\\
&\quad - N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + S(r, F_{a,b})\\
&\leq 2T(r, F_{a,b}) - N_*\left(r, F_{a,b}\right) + S(r, F_{a,b}).
\end{aligned}$$

This completes the lemma.

Lemma 2.5 *[3] Let f be a transcendental meromorphic function of finite order, $c \in \mathbb{C} \setminus \{0\}$ be fixed. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

3 Proof of the Main Theorem

Proof: By the assumption $E(0, F_{a,b}) \subset E(0, \mathcal{L}_n(f, \Delta_c))$, we have

$$N\left(r, \frac{1}{F_{a,b}}\right) \leq N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right). \tag{6}$$

Inequality (6) together with i) of Lemma 2.3, we obtain

$$\begin{aligned} T(r, F_{a,b}) &= T\left(r, \frac{1}{F_{a,b}}\right) + O(1) \\ &\leq m\left(r, \frac{\mathcal{L}_n(f, \Delta_c)}{F_{a,b}}\right) + m\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + N\left(r, \frac{1}{F_{a,b}}\right) + O(1) \\ &\leq m\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + N\left(r, \frac{1}{F_{a,b}}\right) + O(1) \\ &\leq T(r, \mathcal{L}_n(f, \Delta_c)) + S(r, F_{a,b}). \end{aligned} \tag{7}$$

On the other hand by Lemma 2.5

$$T\left(r, \mathcal{L}_n(f, \Delta_c)\right) \leq (k+1)T(r, f) + S(r, f). \tag{8}$$

We observe that $T(r, F_{a,b}) = T(r, af + b) = T(r, f) + O(1)$.
Hence $S(r, F_{a,b}) = S(r, f)$.
We set

$$\frac{\mathcal{L}_n(f, \Delta_c)}{F_{a,b}} = h. \tag{9}$$

The inclusions $E(0, F_{a,b}) \subset E(0, \mathcal{L}_n(f, \Delta_c))$ and $E(\infty, F_{a,b}) \supset E(\infty, \mathcal{L}_n(f, \Delta_c))$ suggest that h is an entire function. Therefore, according to Lemma 2.3, we obtain

$$T(r, h) = m(r, h) + N(r, h) = m(r, h) = m\left(r, \frac{\mathcal{L}_n(f, \Delta_c)}{F_{a,b}}\right) = S(r, f). \tag{10}$$

Next, we consider the following cases for h.
Case 1. If h is not constant.
Since $\mathcal{L}_c^n$ and $F_{a,b}$ share 1CM, we obtain from (9) and (10) that

$$N\left(r, \frac{1}{F_{a,b} - 1}\right) \leq N\left(r, \frac{1}{h-1}\right). \tag{11}$$

By the assumptions $E(0, F_{a,b}) \subset E(0, \mathcal{L}_n(f, \Delta_c))$ and $E(\infty, F_{a,b}) \supset E(\infty, \mathcal{L}_n(f, \Delta_c))$, we notice that

$$N\left(r, \mathcal{L}_n(f, \Delta_c)\right) - N(r, F_{a,b}) \leq 0, \quad N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) \leq 0 \tag{12}$$

By applying Lemma 2.4, combining with (11) and (12), we have

$$\begin{aligned} T(r, F_{a,b}) &= N\left(r, \frac{1}{F_{a,b}-1}\right) + N\left(r, \mathcal{L}_n(f, \Delta_c)\right) - N(r, F_{a,b}) + N\left(r, \frac{1}{F_{a,b}}\right) \\ &\quad - N\left(r, \frac{1}{\mathcal{L}_n(f, \Delta_c)}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{F_{a,b}-1}\right) + S(r, f) \leq S(r, f), \end{aligned}$$

which is a contradiction.
Case 2. If h is constant.
Subcase 1. Assume $h \neq 1$. In this scenario, 1 becomes a Picard exceptional value for $F_{a,b}$ based on the assumption that $F_{a,b}$ and $\mathcal{L}_n(f,\ \Delta_c)$ share 1 CM. Consequently, $N\left(r, \frac{1}{F_{a,b}-1}\right) = O(1)$. Following the same argument as in case 1, we encounter a contradiction.
Subcase 2. If $h = 1$, then the condition holds, that is $\mathcal{L}_n(f, \Delta_c) \equiv F_{a,b}$.

OPEN PROBLEM. Can the result obtained in this paper be generalized by replacing $F_{a,\,b}$ with $P_n(f) = a_n f^n + \cdots + a_1 f + a_0$?

ACKNOWLEDGEMENTS. We are deeply grateful for the referee's constructive comments and insightful suggestions on improving the paper's presentation.

References

[1] Banerjee, A.; Basir Ahamed, M. Results on meromorphic function sharing two sets with its linear c-difference operator.Izv. Nats. Akad. Nauk Armenii Mat.55(2020), no.3, 3–20; translation in J. Contemp. Math. Anal.55(2020), no.3, 143–155.

[2] Chen, Zong-Xuan; Yi, Hong-Xun On sharing values of meromorphic functions and their differences.Results Math.63(2013), no.1-2, 557–565.

[3] Y.-M. Chiang and S.-J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, Ramanujan J. **16** (2008), no. 1, 105–129.

[4] *Lü*, Feng; *Lü*, Weiran Meromorphic functions sharing three values with their difference operators.Comput. Methods Funct. Theory17(2017), no.3, 395–403.

[5] Halburd, R. G.; Korhonen, R. J. Nevanlinna theory for the difference operator.Ann. Acad. Sci. Fenn. Math.31(2006), no.2, 463–478.

[6] MokHoN'Ko, A. Z. "The Nevanlinna characteristics of certain meromorphic functions." Theory of functions, functional analysis and their applications 14 (1971): 83-87.

[7] Waghamore, Harina P., and B. E. Manjunath. "Difference monomial and its shift sharing a polynomial." Palestine Journal of Mathematics 12.2 (2023).