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Perfect Fluid Spacetime with Torse-Forming Vector Field and Gradient Soliton

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Abstract

The purpose of the present paper is to study the Ricci bounds of perfect fluid spacetime equipped with torse-forming vector field. We have analysed some basic geometrical properties related to the spacetime. We have calculated the Ricci operator and shown non-existence of the matter. Finally the characterizations have been made with gradient ρ-Einstein, gradient m-quasi Einstein and gradient (m, ρ)-quasi Einstein solitons on a perfect fluid spacetime.

Keywords: *Perfect fluid spacetimes, gradient Ricci solitons, m-quasi Einstein solitons, (m,ρ)-quasi Einstein solitons.*

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1 Introduction

In general relativity, the spacetime and cosmology can be modeled as a n-dimensional Lorentzian manifold whose Lorentzian metric is with the signature (-,+,+,...,+). K. L. Duggal[6] showed that a connected smooth manifold (M,g) of dimension >2 with a Lorentzian metric g of signature (-,+,+,....,+) which has a global timelike vector field is referred as a spacetime. A fluid with no heat conduction and no viscosity or looks isotropic is said to be perfect fluid. A Lorentzian manifold M is named as a perfect fluid spacetime(PFS) if the Ricci tensor S of M satisfies

$$S = ag + bA \otimes A, \tag{1}$$

in which a, b are scalar fields, A is a 1-form defined as $A(X) = g(X, \xi)$ for all X, where ξ is a finite timelike vector field [5]. The physical and geometrical representation of PFS on Ricci solitons were extensively studied in the last decade by several authors such as De [4], De[5] and many more.
Hamilton[7] introduced the notion of Ricci flow which describes certain partial differential equation for a Riemannian metric $\frac{\partial}{\partial t}g(t) = -2S(t), t \geq 0, g(0) = g$, where g is the Riemannian metric and S denotes (0,2)-symmetric Ricci tensor. Diligently, Ricci soliton on Riemannian manifold was broadly studied by several authors such as Sharma[9], Bejan and Crasmarenu[2]. A Ricci soliton on a Riemannian manifold (M, g) is a triple (g, V, λ) which satisfies the condition

$$L_V g + 2S + 2\lambda g = 0, \tag{2}$$

where, S is the Ricci tensor, L_V denote the lie-derivative along the vector field V on M and λ is a cosmological constant. The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively.
If the soliton vector V is the gradient of smooth function $-f$, that is, $V = -Df$, then equation(2) is of the form,

$$Hessf - S - \lambda g = 0, \tag{3}$$

where Hess is the Hessian and D is the gradient operator. The metric is named as a gradient Ricci Soliton if it obeys equation (3). The smooth function $-f$ is called the potential function of the gradient Ricci Soliton.
The concept of gradient ρ-Einstein soliton is introduced and studied by Catino and Mazzieri [3], which is a special case of the Ricci Soliton.The metric $'g'$ on a Riemannian manifold M is a gradient ρ- Einstein soliton if there is a smooth function f such that the metric g satisfies the equation

$$S + Hessf = (\rho r + \lambda)g = \beta g \tag{4}$$

for some constant $\lambda \in R$, where r is the scalar curvature. In recent years there has been made an extensive study of Riemannian manifolds endowed with gradient ρ-Einstein soliton.
A (m,ρ)-quasi Einstein soliton is also a generalization of Ricci soliton which was studied by Shin[10]. If the potential vector field V is the gradient of a smooth function $f \in C^\infty(M)$ then the (m,ρ)-quasi Einstein manifold is called as a gradient (m, ρ)-quasi Einstein manifold. So we have ,

$$Hessf + S = \frac{1}{m} df \otimes df + (\lambda + \rho r)g. \tag{5}$$

A (pseudo)- Riemannian manifold M furnished with the semi-Riemannian metric $'g'$ is named a gradient m-quasi Einstein metric [1] if, for a constant η and a smooth function $f : M \to R$, we have

$$S + Hessf = \frac{1}{m} df \otimes df + \lambda g, \tag{6}$$

in which $0 < m \leq \infty$ is an integer and $\otimes$ denotes tensor product.
In this setting, f indicates the m-quasi Einstein potential function. In this case, the Bakry-Emery Ricci tensor [11] $S + Hessf = \frac{1}{m} df \otimes df$ is proportional to the metric g and η is a constant.
If $m = \infty$, (6) represents a gradient Ricci Soliton. If $m = \infty$ and η is a smooth function, then the metric represents almost gradient Ricci Soliton on Einstein product manifold with non-empty base.
The organisation of the paper is as follows: in section 2 perfect fluid spacetime with torse-forming vector field is studied. In section 3, the study was made with Ricci Soliton on a perfect fluid spacetime, calculated the Ricci operator and shown the non-existence of matter and sign of the pressure. Section 4 is devoted to the study of gradient solitons by showing the function $'f'$ is invariant under the velocity vector field ξ and characterizations are made by considering gradient ρ-Einstein, gradient m-quasi Einstein and gradient (m, ρ)-quasi Einstein solitons on a perfect fluid spacetime.

2 Preliminaries

In a perfect fluid spacetime, ξ is the unit timelike vector field also called as the velocity vector field of the fluid which satisfies,

$$g(U, \xi) = A(U), g(\xi, \xi) = A(\xi) = -1, \tag{7}$$

where U belongs to $\chi(M), \chi(M)$ denotes the collection of all C^∞ vector fields of M and A is a non-zero 1-form.
Applying covariant derivative on (7), we get $g(\nabla_U \xi, \xi) = 0, (\nabla_U A)(\xi) = 0$, where ∇ is the levi-civita connection.
The Einstein's field equation is of the form as,

$$S - \frac{r}{2} g = \kappa T, \tag{8}$$

where T is the energy momentum tensor and κ is the gravitational constant.
The energy mementum tensor in a PFS is in the following form

$$T = (\sigma + p) A \otimes A + pg \tag{9}$$

where $'\sigma'$ and $'p'$ denote the energy density and isotropic pressure of a perfect fluid spacetime respectively.
Contracting (1) we get,

$$r = na - b. \tag{10}$$

Now, from equations (1),(8) and (9) we get,

$$b = k(p + \sigma) \quad and \quad a = \frac{k(p - \sigma)}{2 - n}. \tag{11}$$

On substitution of the values for 'a' and 'b' in equation (10) we get,

$$r = (\frac{k(p - \sigma)}{2 - n})n - k(p + \sigma). \tag{12}$$

Definition 2.1 *A vector field ξ is called torse-forming if it satisfies*

$$\nabla_U \xi = U + A(U)\xi, \tag{13}$$

for any $X \in \chi(M)$ and A is a 1-form.

The use of (1) and (7) results in $S(U, \xi) = (a - b)A(U)$, where, $a - b$ is an eigenvalue of S.

3 Perfect Fluid Spacetime

In this section we prove some basic results and find the bounds of Ricci tensor in a perfect fluid spacetime.

Theorem 3.1 *Let M(g,ξ,η, V) be a perfect fluid spacetime with g has a Ricci Soliton. If V is a torse-forming vector field then M represents an η-Einstein manifold with constant scalar curvature.*

Proof 3.1 *Replacing V by ξ in (2), and using (13), we obtain*

$$S(U, V) = -(\lambda + 1)g(U, V) - A(U)A(V). \tag{14}$$

Taking orthonormal basis $\{e_i\} : 1 \leq i \leq n - 1$ and summing over $1 \leq i \leq n - 1$ in (14), we obtain

$$r = 1 - (\lambda + 1)(n - 1). \tag{15}$$

By making use of equation (12) in (15), we obtain

$$\lambda = \frac{k(p + \sigma) - n(\frac{k(p-\sigma)}{(2-n)} + 1) + 2}{(n - 1)}. \tag{16}$$

Thus from equations (14) and (15) the result follows.

Proposition 3.2 *In a perfect fluid spacetime with torse-forming vector field ξ, the square of length of the Ricci operator is $\|Q\|^2 = \frac{k^2(p-\sigma)^2 n}{(2-n)^2} + \frac{k^2(p+\sigma)}{(2-n)}[\sigma(4 - n) - np]$.*

Proof 3.2 *Taking X=QX in (1), we get*

$$S(QX, Y) = aS(X, Y) + bS(X, \xi)A(Y). \tag{17}$$

On contraction of the above equation over X and Y, we get

$$S^2(X, X) = \|Q\|^2 = ar + bS(\xi, \xi). \tag{18}$$

From (1), we have

$$S(\xi, \xi) = b - a = k(p + \sigma) - \frac{k(p - \sigma)}{2 - n}, \tag{19}$$

In view of equation (19), (18) becomes,

$$\|Q\|^2 = \frac{k(p - \sigma)r}{(2 - n)} + \frac{k^2(p + \sigma)}{(2 - n)}[p(1 - n) + \sigma(3 - n)]. \tag{20}$$

By virtue of equation(10), the above equation reduces to

$$\|Q\|^2 = \frac{k^2(p - \sigma)^2 n}{(2 - n)^2} + \frac{k^2(p + \sigma)}{(2 - n)}[\sigma(4 - n) - np]. \tag{21}$$

Theorem 3.3 *If in a perfect fluid spacetime without cosmological constant, square of the length of the Ricci operator is $\frac{1}{3}r^2$, then the spacetime does not contain pure matter and the pressure of the fluid is negative.*

Proof 3.3 *Let us assume that, the length of the Ricci operator to be $\frac{1}{3}r^2$, where 'r' is the scalar curvature of the spacetime. Then equation (21) becomes,*

$$\frac{1}{3}r^2 = \frac{k^2(p - \sigma)^2 n}{(2 - n)^2} + \frac{k^2(p + \sigma)}{(2 - n)}[\sigma(4 - n) - np]. \tag{22}$$

By making use of equation (12) the above equation yields,

$$\frac{k^2 n(n - 3)(p - \sigma)^2}{2 - n} + \frac{k^2(p + \sigma)}{3(2 - n)}(3\sigma(n - 3) + p(3n - 1)) = 0. \tag{23}$$

Since $(p - \sigma) \neq 0$ and $k \neq 0$, we have $\sigma = 0$, which is not possible, as when matter exists, σ is always greater than 0. Thus the spacetime does not contain pure matter.
Further we determine the sign of the pressure by considering equation (12) for spacetime without pure matter,

$$p = -\frac{(2 - n)r}{2k}. \tag{24}$$

Hence the result follows.

Theorem 3.4 *The square of the length of the Ricci operator in perfect fluid spacetime without pure matter satisfying Einstein's gravitational equation with cosmological constant is* $\|Q\|^2 = \frac{2(k\rho-\lambda)}{(3-n)^2}[(n-1)(2+k\rho)(k\rho-\lambda)+k\rho]+\frac{(1-n)}{(2-n)}(k\rho)^2$.

Proof 3.4 *Einstein's gravitational equation for a perfect fluid motion, is as follows:*

$$S(X,Y) + \left(\lambda - \frac{r}{2}\right) g(X,Y) = \kappa T(X,Y). \tag{25}$$

In view of equations (9) and (25), we get

$$S(X,Y) = \left(k\rho - (\lambda - \frac{r}{2})\right) g(X,Y) + k(\sigma+\rho)A(X)A(Y). \tag{26}$$

Now, at each point of the manifold, we take orthonormal basis $\{e_i\} : 1 \leq i \leq n-1$. *By substituting* $X = Y = e_i$ *and summing over* $1 \leq i \leq n-1$ *in (26), we obtain*

$$r = (n-1)\left(k\rho - (\lambda - \frac{r}{2})\right) + k(\sigma+\rho). \tag{27}$$

Rewritting 'r' from above equation, we get

$$r = \frac{2(n-1)(k\rho-\lambda)}{(3-n)} + \frac{2k(\sigma+\rho)}{(3-n)}. \tag{28}$$

Substituting the value of 'r' in equation (26), we obtain

$$S(X,Y) = \left((k\rho-\lambda)(1+\frac{(n-1)}{(3-n)}) + \frac{k(\sigma+\rho)}{(3-n)}\right) g(X,Y) + k(\sigma+\rho)A(X)A(Y). \tag{29}$$

Replacing X by QX in equation (26), we get

$$\begin{aligned} S(QX,Y) = &\left((k\rho-\lambda)(1+\frac{(n-1)}{(3-n)}) + \frac{k(\sigma+\rho)}{(3-n)}\right) g(QX,Y) \\ &+k(\sigma+\rho)A(QX)A(Y). \end{aligned} \tag{30}$$

On contraction, the above equation reduces to,

$$\|Q\|^2 = \frac{2(k\rho-\lambda)}{(3-n)^2}(2+k(\sigma+\rho))((n-1)(k\rho-\lambda)+k(\sigma+\rho)) + (b-a)k(\sigma+\rho). \tag{31}$$

On substitution of the values for 'a' and 'b' we get,

$$\begin{aligned} \|Q\|^2 = &\frac{2(k\rho-\lambda)}{(3-n)^2}(2+k(\sigma+\rho))((n-1)(k\rho-\lambda)+k(\sigma+\rho)) \\ &+\frac{k(\sigma+\rho)}{(2-n)}(k(2-n)(\rho+\sigma) - k(\rho-\sigma)). \end{aligned} \tag{32}$$

Putting σ*=0, the above equation reduces to,*

$$\|Q\|^2 = \frac{2(k\rho-\lambda)}{(3-n)^2}[(n-1)(2+k\rho)(k\rho-\lambda)+k\rho] + \frac{(1-n)}{(2-n)}(k\rho)^2. \tag{33}$$

Thus the result follows.

4 Perfect fluid spacetime with gradient solitons

In this section we show that under certain conditions perfect fluid spacetime represents dark energy. The investigations were made on dark energy[5] in which the Lorentzian metric of a PFS, M denotes the dark energy. Further in[4] K. De and U. C. De has charecterized perfect fluid spacetime whose Lorentzian metric equipped with gradient m-quasi Einstein solitons represents dark energy.

Theorem 4.1 *Let M be a perfect fluid spacetime adorned with torse forming vector field with* $\xi a = -bf$. *Then,*

1. *The state equation in a perfect fluid spacetime is governed by* $p = -\frac{\sigma(3-n)}{(1-n)}$.
2. f *is invariant under the velocity vector* ξ.

Proof 4.1 *Suppose g is the the gradient Ricci soliton on a perfect fluid spacetime (M, g, ξ,η, V) admitting a torse-forming vector field V.*
We have,

$$(\nabla_V A)(U) = \nabla_V A(U) - A(\nabla_V U) \tag{34}$$

$$= \nabla_V g(U, \xi) - g(\nabla_V U, \xi) \tag{35}$$

$$g(U, \nabla_V \xi) = g(U, V) + A(U)A(V) \tag{36}$$

forall $U, V \in \chi(M)$ *equation (1) imply that*

$$QU = aU + bA(U)\xi, \forall\ U \in \chi(M). \tag{37}$$

Suppose the gradient Ricci soliton for some smooth function $-f$. *Then from (2), we get*

$$\nabla_U Df = QU + \lambda U, \tag{38}$$

for all $U \in \chi(M)$.
We have

$$R(U, V) = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df. \tag{39}$$

By using the above equation in (35), we get

$$R(U, V)Df = (\nabla_U Q)V - (\nabla_V Q)U. \tag{40}$$

Taking Covariant derivative of (34) and utilizing (36), we obtain

$$\begin{aligned}(\nabla_U Q)V = (Ua)V + (Ub)A(V)\xi + bg(U, V)\xi + bA(U)A(V)\xi \\ +bA(V)U + bA(V)A(U)\xi.\end{aligned} \tag{41}$$

By making use of the above equation in (40), we get

$$R(U,V)Df = (Ua)V - (Va)U + (Ub)A(V)\xi - (Vb)A(U)\xi + b(A(V)U - A(U)V). \tag{42}$$

Taking orthonormal basis $\{e_i\} : 1 \leq i \leq n-1$ *and contracting, the above equation gives*

$$S(U,Df) = (1-n)(Ua) + (Ub) + (\xi b)A(U) + (n-1)bA(U). \tag{43}$$

Replacing U *by* ξ *in above equation, we get*

$$S(\xi,Df) = (1-n)(\xi a + b). \tag{44}$$

From (1), we have

$$S(V,Df) = a(Vf) + bA(V)(\xi f). \tag{45}$$

Putting $V{=}\xi$ *in the above equation, we obtain*

$$S(\xi,Df) = a(\xi f) - b(\xi f). \tag{46}$$

Comparing (44) and (46) we get,

$$(a-b)(\xi f) = (1-n)(\xi a + b). \tag{47}$$

Suppose $\xi a = -b$. *Then (47) becomes*

$$(a-b)(\xi f) = 0. \tag{48}$$

From previous equation it is illustrated that either $a{=}b$ *or* $\xi f{=}0$.
So we can generalize two cases:
Case 1: If $a = b$ *and* $\xi f \neq 0$ *then from equation (11) it is illustrated that*

$$p = -\frac{\sigma(3-n)}{(1-n)}, \tag{49}$$

which shows the form of the state equation in perfect fluid spacetime.
Case 2: If $\xi f = 0$ *and* $a \neq b$, *then it is seen that* f *is invariant under the vector field* ξ. *Thus the result follows.*

Theorem 4.2 *Let (M,g,V) be a perfect fluid spacetime with torse-forming vector field V be such that*

1. *g is a gradient m-quasi Einstein soliton,* $m{\neq}0$ *or*

2. *g is a gradient (m-*ρ*) quasi Einstein soliton,* $m{\neq}0$ *or*

3. g is gradient ρ-Einstein soliton,

then M denotes dark energy provided f, b and a are invariant under ξ.

Proof 4.2 *Let (M,g) be Perfect fluid spacetime admitting torse-forming vector field. Suppose g is a gradient m-quasi Einstein metric, (6) can be expressed as,*

$$\nabla_U Df + QU = \frac{1}{m} g(U, Df)Df + \lambda U. \tag{50}$$

Applying covariant derivative in the above equation we get,

$$\nabla_V \nabla_U Df = -\nabla_V QU + \frac{1}{m}\nabla_V g(U, Df)Df + \frac{1}{m} g(U, Df)\nabla_V Df + \lambda \nabla_V U. \tag{51}$$

Interchanging U and V in the above equation, we obtain

$$\nabla_U \nabla_V Df = -\nabla_U QV + \frac{1}{m}\nabla_U g(V, Df)Df + \frac{1}{m} g(V, Df)\nabla_U Df + \lambda \nabla_U V, \tag{52}$$

and also, we have

$$\nabla_{[U,V]} Df = -Q[U, V] + \frac{1}{m} g([U, V], Df)Df + \lambda [U, V]. \tag{53}$$

By making use of equations (51),(52)and (53) in $R(U, V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df$ reduces to,

$$\begin{aligned} R(U, V)Df = (\nabla_V Q)U - (\nabla_U Q)V + \frac{\lambda}{m}((Vf)U - (Uf)V) \\ + \frac{1}{m}((Uf)QV - (Vf)QU). \end{aligned} \tag{54}$$

Utilizing (41) and (37) in the above equation we get,

$$\begin{aligned} R(U, V)Df = (Va)U - (Ua)V + ((Vb)A(U) - (Ub)A(V))\xi \\ + b(A(U)V - A(V)U) + \frac{\lambda}{m}((Vf)U - (Uf)V) \\ + \frac{1}{m}((Uf)(aV + bA(V)\xi) - (Vf)(aU + bA(U)\xi)). \end{aligned} \tag{55}$$

Taking orthonormal basis and by contraction in the above equation we get,

$$\begin{aligned} S(U, Df) = (Ua)(n-1) - (\xi b)A(U) - (Ub) - (n-1)bA(U) \\ + \frac{\lambda}{m}(n-1)(Uf) - \frac{1}{m}(a(Uf)(n-1) - b(Uf) - b(\xi f)A(U)). \end{aligned} \tag{56}$$

Putting U=ξ in the above equation we get,

$$S(\xi, Df) = (n-1)(\xi a + b) + \frac{(n-1)}{m}(\lambda - a)\xi f. \tag{57}$$

Equating $S(\xi, Df)$ of (46) and the above equation we get,

$$(\xi f)[m(a-b) + (n-1)(a-\lambda)] = (n-1)m(\xi a + b). \tag{58}$$

If f and a are invariant under the velocity vector field ξ, then we get from above equation that $b = 0$. Thus this prooves first result.
Next suppose Perfect fluid spacetime with torse forming vector field admitting (m-ρ)-quasi Einstein metric. Equation (5) can be expressed as,

$$\nabla_U Df + QU = \frac{1}{m} g(U, Df)Df + \beta U. \tag{59}$$

By applying covariant derivative in the above equation we get,

$$\nabla_V \nabla_U Df = -\nabla_V QU + \frac{1}{m}\nabla_V g(U, Df)Df + \frac{1}{m} g(U, Df)\nabla_V Df + \beta \nabla_V U. \tag{60}$$

Interchanging U and V in the above equation we get,

$$\nabla_U \nabla_V Df = -\nabla_U QV + \frac{1}{m}\nabla_U g(V, Df)Df + \frac{1}{m} g(V, Df)\nabla_U Df + \beta \nabla_U V, \tag{61}$$

and also we have,

$$\nabla_{[U,V]} Df = -Q[U,V] + \frac{1}{m} g([U,V], Df)Df + \beta[U,V]. \tag{62}$$

By making use of equations (60),(61) and (62) in $R(U,V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df$, we find

$$\begin{aligned} R(U,V)Df &= (Va)U + (Vb)A(U)\xi + bA(U)V - (Ua)V - (Ub)A(V)\xi \\ &\quad - bA(V)U + \frac{\beta}{m}((Vf)U - (Uf)V) + \frac{1}{m}((Uf)QV - (Vf)QU). \end{aligned} \tag{63}$$

Taking orthonormal basis and by contraction of the above equation we get,

$$\begin{aligned} S(U, Df) &= -(\xi b)A(U) - (n-1)bA(U) + (n-1)(Ua) - (Ub) \\ &\quad + (n-1)\frac{\beta}{m}(Uf) - \frac{1}{m}((Uf)(a(n-1) - b) - b(\xi f)A(U)). \end{aligned} \tag{64}$$

Replacing U by ξ we obtain,

$$S(\xi, Df) = (n-1)b + (n-1)(\xi a) + \frac{\beta}{m}(n-1)(\xi f) - \frac{1}{m}(n-1)a(\xi f). \tag{65}$$

Equating $S(\xi, Df)$ of (46) and the above equation we get,

$$\xi f(m(a-b)+(n-1)(a-\beta)) = (n-1)m(\xi a+b). \tag{66}$$

We assume that f, b and a are invariant under the velocity vector field ξ, then we get from the above equation that b=0. Thus the result follows for second case.

We take perfect fluid spacetime as a gradient ρ- Einstein Soliton with V as a torse forming vector field. Then, (4) can be expressed as

$$\nabla_U Df + QU = \beta U. \tag{67}$$

By applying covariant derivative in the above equation we get,

$$\nabla_V \nabla_U Df = -\nabla_V QU + \beta \nabla_V U. \tag{68}$$

Replacing U By V in the above equation we get,

$$\nabla_U \nabla_V Df = -\nabla_U QV + \beta \nabla_U V. \tag{69}$$

and also we have,

$$\nabla_{[U,V]} Df = -Q[U,V] + \beta[U,V]. \tag{70}$$

By making use of (68), (69) and 70) in $R(U,V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df$ we get,

$$\begin{aligned} R(U,V)Df = (Va)U + (Vb)A(U)\xi + bA(U)V - (Ua)V \\ -(Ub)A(V)\xi - bA(V)U. \end{aligned} \tag{71}$$

Taking orthonormal frame field over $i = 1 \cdots n-1$ and by contraction of the above equation we get,

$$S(U, Df) = (n-1)Ua - Ub - (\xi b)A(U) - (n-1)bA(U). \tag{72}$$

Replacing U by ξ in the above equation we get,

$$S(\xi, Df) = (n-1)(\xi a + b). \tag{73}$$

Equating $S(\xi, Df)$ of (46) and the above equation we get,

$$(\xi f)(a-b) = (n-1)(\xi a + b). \tag{74}$$

For constants f, b and a along ξ, we get b=0. Hence the result is obtained for third case.

5 Open Problem

In this paper we have examined some basic geometrical properties related to perfect fluid spacetime equipped with torse forming vector field using Ricci soliton. This type of work can be done by making use of Yamabe soliton in which the results will be more itriguing.

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