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On potential wells for a nonlinear higher-order hyperbolic-type equation

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Abstract

In this work, we deal with the higher-order hyperbolic-type equation. Firstly, we prove the global existence and blow up of solutions with the subcritical initial energy. Later, we prove the global existence and blow up of solutions with the critical initial energy.

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1 Introduction

In this work, we are concerned the following higher-order hyperbolic type equation

$$\begin{cases} z_{tt} + \mathcal{A}z = |z|^{q-1} z, & x \in \Omega,\ t > 0, \\ \frac{\partial^i z(x,t)}{\partial v^i} = 0,\ i = 0, 1, ..., m-1, & x \in \partial\Omega,\ t \geq 0, \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x), & x \in \Omega \end{cases} \tag{1}$$

where $\mathcal{A} = (-\Delta)^m,\ m \geq 1$ is a natural number, Ω is a bounded domain with smooth boundary $\partial\Omega$ in $R^n\ (n \geq 1)$, so that the divergence theorem can be applied, $\frac{\partial^i z(x,t)}{\partial v^i}$ denotes the $i-$order normal derivation of z and v is unit outward normal vector on $\partial\Omega$. The power index q of the source term satisfies.

$$\begin{cases} 1 < q < \infty, & n \le 2m, \\ 1 < q < \frac{n}{n-2m}, & n > 2m. \end{cases}$$

In [12], Vitillaro considered the local existence, global existence and blow up of solutions with nonlinear boundary condition for the following wave equation

$$z_{tt} - \Delta z = |z|^{q-1} z.$$

In [14], Yacheng studied the potential wells of solutions for the following wave equation

$$z_{tt} - \Delta z = |z|^{q-1} z.$$

Later, Liu and Li [4] studied the global existence and blow up of solutions with the subcritical and critical initial energy for the same equation.

After that many authors [1, 2, 3, 9, 10, 11] considered the existence and blow up of solutions of the problem for various nonlinear PDE by using the potential well method.

We focus on a family of new potential wells and their applications to higher-order hyperbolic-type equation (1). The potential well was introduced by Payne and Sattinger [6] and Sattinger [7]. The main purpose of this work is to construct a family of new potential wells and their outside sets by modifying the depths of the potential wells for the higher-order hyperbolic-type equation inspired by [4, 13].

In several mathematical models, we face higher-order partial differential equations (PDE). For example, it can be found in fluid dynamics, electromagnetism, biology, mechanics and image processing, here 3-dimensional problems are represented on the surfaces, for instance in the case of thin geometries, modeled as membranes, plates or shells, depending on the structure of the original domain. This leads to defining surface partial differential equations which often involve high-order differential operators [8].

The outline of this work is as follows: In part 2, we define the potential wells for our problem. In part 3, we prove the global existence and blow up of solutions with the subcritical initial energy ($E(0) < d(\delta)$). In part 4, we prove the global existence and blow up of solutions with the critical initial energy ($E(0) = d(\delta)$).

In this work, we denote

$$\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)},\ \|\cdot\| = \|\cdot\|_2,\ (z, v) = \int_\Omega zv dx.$$

2 Setup of potential wells

In this section, we define the potential wells for our problem.

2.1 Potential wells family and their depths

We define the energy functional as

$$E\left(t\right)=\frac{1}{2}\|z_t\|^2+\frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}z\|^2-\frac{1}{q+1}\|z\|_{q+1}^{q+1},$$

which satisfies the energy identity

$$E\left(t\right)=E\left(0\right)$$

for all $t\geq 0$. Also, we define the auxiliary functional

$$J_\delta\left(z\right)=\frac{\delta}{2}\|\mathcal{A}^{\frac{1}{2}}z\|^2-\frac{1}{q+1}\|z\|_{q+1}^{q+1},\ 0<\delta\leq 1.$$

Now, we are able to define the depths of the potential wells

$$d\left(\delta\right)=\max_{y\in[0,\infty)}g_\delta\left(y\right),$$

here

$$y=\|\mathcal{A}^{\frac{1}{2}}z\|,$$
$$g_\delta\left(y\right)=\frac{\delta}{2}y^2-\frac{1}{q+1}c^{q+1}y^{q+1}$$

and c is the best constant for the Sobolev embedding from $H_0^m\left(\Omega\right)$ into $L^{q+1}\left(\Omega\right)$.

Let $g_\delta'\left(y\right)=0$, then

$$y_\delta=\delta^{\frac{1}{q-1}}c^{-\left(\frac{q+1}{q-1}\right)}. \tag{2}$$

From here

$$d\left(\delta\right)=g_\delta\left(y_\delta\right)=\frac{q-1}{2\left(q+1\right)}\delta^{\frac{q+1}{q-1}}c^{-2\left(\frac{q+1}{q-1}\right)}. \tag{3}$$

By using the (2) and (3), we have

$$y_\delta=\left[\frac{d\left(\delta\right)}{\delta}\frac{2\left(q+1\right)}{q-1}\right]^{\frac{1}{2}}.$$

Like this, we can define a family of potential wells

$$W_\delta=\left\{z\in H_0^m\left(\Omega\right)\mid\|\mathcal{A}^{\frac{1}{2}}z\|<\left[2\left(\frac{q+1}{q-1}\right)\frac{d\left(\delta\right)}{\delta}\right]^{\frac{1}{2}}\right\},$$

and their outside sets

$$V_\delta=\left\{z\in H_0^m\left(\Omega\right)\mid\|\mathcal{A}^{\frac{1}{2}}z\|>\left[2\left(\frac{q+1}{q-1}\right)\frac{d\left(\delta\right)}{\delta}\right]^{\frac{1}{2}}\right\}.$$

It is obvious

$$\partial W_\delta=\partial V_\delta=\left\{z\in H_0^m\left(\Omega\right)\mid\|\mathcal{A}^{\frac{1}{2}}z\|=\left[2\left(\frac{q+1}{q-1}\right)\frac{d\left(\delta\right)}{\delta}\right]^{\frac{1}{2}}\right\}.$$

Lemma 2.1 *Assume that* $z \in H_0^m(\Omega)$.

i) If $z \in W_\delta$ *and* $\|\mathcal{A}^{\frac{1}{2}} z\| \neq 0$, *then* $\delta \|\mathcal{A}^{\frac{1}{2}} z\|^2 > \|z\|_{q+1}^{q+1}$.

ii) If $z \in \partial W_\delta$, *then* $\delta \|\mathcal{A}^{\frac{1}{2}} z\|^2 \geq \|z\|_{q+1}^{q+1}$.

iii) If $\delta \|\mathcal{A}^{\frac{1}{2}} z\|^2 < \|z\|_{q+1}^{q+1}$, *then* $z \in V_\delta$.

iv) If $\delta \|\mathcal{A}^{\frac{1}{2}} z\|^2 = \|z\|_{q+1}^{q+1}$ *and* $\|\mathcal{A}^{\frac{1}{2}} z\| \neq 0$, *then* $z \in H_0^m(\Omega) \setminus W_\delta = V_\delta \cup \partial V_\delta$.

Proof 2.2 *i) Since* $z \in W_\delta$, *we obtain*

$$\|\mathcal{A}^{\frac{1}{2}} z\| < \left[2 \left(\frac{q+1}{q-1} \right) \frac{d(\delta)}{\delta} \right]^{\frac{1}{2}},$$

which, together with (3), we have

$$\|\mathcal{A}^{\frac{1}{2}} z\| < \delta^{\frac{1}{q-1}} c^{-\frac{q+1}{q-1}}.$$

From here

$$\delta > c^{q+1} \|\mathcal{A}^{\frac{1}{2}} u\|^{q-1}.$$

Since $\|\mathcal{A}^{\frac{1}{2}} z\| \neq 0$, *multiplying the above inequality by* $\|\mathcal{A}^{\frac{1}{2}} z\|^2$, *we get*

$$\delta \|\mathcal{A}^{\frac{1}{2}} z\|^2 > c^{q+1} \|\mathcal{A}^{\frac{1}{2}} z\|^{q+1},$$

and so

$$\|z\|_{q+1}^{q+1} < \delta \|\mathcal{A}^{\frac{1}{2}} z\|^2.$$

ii) From $z \in \partial W_\delta$ *we get*

$$\|\mathcal{A}^{\frac{1}{2}} z\| = \left[\frac{2(q+1)}{q-1} \frac{d(\delta)}{\delta} \right]^{\frac{1}{2}}.$$

Similarly to (i), we have

$$\|z\|_{q+1}^{q+1} \leq \delta \|\mathcal{A}^{\frac{1}{2}} z\|^2.$$

iii) Taking into account $\|\mathcal{A}^{\frac{1}{2}} z\| \neq 0$, *we have*

$$\begin{aligned} \delta \|\mathcal{A}^{\frac{1}{2}} z\|^2 &< \|z\|_{q+1}^{q+1} \\ &\leq c^{q+1} \|\mathcal{A}^{\frac{1}{2}} z\|_2^{q+1}, \end{aligned}$$

thus

$$\delta < c^{q+1} \|\mathcal{A}^{\frac{1}{2}} z\|_2^{q-1}.$$

We further get

$$\|\mathcal{A}^{\frac{1}{2}} z\| > \delta^{\frac{1}{q-1}} c^{-\frac{q+1}{q-1}}.$$

Combining this with (3), we get

$$\|\mathcal{A}^{\frac{1}{2}}z\| > \left[\frac{2\,(q+1)}{q-1}\frac{d\,(\delta)}{\delta}\right]^{\frac{1}{2}}.$$

Hence $z \in V_\delta$.

iv) By the proof of (iii) we know that $\delta\|\mathcal{A}^{\frac{1}{2}}z\|^2 = \|z\|_{q+1}^{q+1}$ *and* $\|\mathcal{A}^{\frac{1}{2}}z\|^2 \neq 0$ *imply*

$$\|\mathcal{A}^{\frac{1}{2}}z\| \geq \left[\frac{2\,(q+1)}{q-1}\frac{d\,(\delta)}{\delta}\right]^{\frac{1}{2}}.$$

Hence $z \in H_0^m(\Omega) \setminus W_\delta = V_\delta \cup \partial V_\delta$.

2.2 Invarience of the potential wells and their outside sets

In this part, we show that W_δ and V_δ are both invariant under the flow of problem (1) with the subcritical inital energy.

Definition 2.3 *Function* $z = z(x,t)$ *is called a weak solution of problem (1) over* $\Omega \times [0,T)$, *if* $z \in L^\infty(0,T;H_0^m(\Omega))$, $z_t \in L^\infty(0,T;L^2(\Omega))$, *satisfying*

i) for all $v \in H_0^m(\Omega)$ *and a.e.* $t \in [0,T)$

$$(z_t(t), v) + \int_0^t \left(\mathcal{A}^{\frac{1}{2}}z(t), \mathcal{A}^{\frac{1}{2}}v\right) d\tau = (z_1, v) + \int_0^t \left(|z(\tau)|^{q-1}z(\tau), v\right) d\tau, \quad (4)$$

ii) $z(0) = z_0 \in H_0^m(\Omega)$, $z_t(0) = z_1 \in L^2(\Omega)$.

Theorem 2.4 *Suppose that* z *be a solution of problem (1) on* $\Omega \times [0,T)$. *Assume further that* $z \in H_0^m(\Omega)$, $z_1 \in L^2(\Omega)$ *and* $0 < E(0) < d(\delta)$.

i) If $z_0 \in W_\delta$, *then* $z(t) \in W_\delta$ *for all* $t \in [0,T)$.

ii) If $z_0 \in V_\delta$, *then* $z(t) \in V_\delta$ *for all* $t \in [0,T)$.

Proof 2.5 *i) Suppose that* $z(t) \notin W_\delta$ *for some* $0 < t < T$. *Then we see from* $z_0 \in W_\delta$ *that there exists the first time* $0 < t_0 < T$ *such that* $z(t_0) \in \partial W_\delta$.

Thus

$$\|\mathcal{A}^{\frac{1}{2}}z(t_0)\| = \left[\frac{2\,(q+1)}{q-1}\frac{d\,(\delta)}{\delta}\right]^{\frac{1}{2}}.$$

Consequently, from item (ii) of Lemma 1 that

$$
\begin{aligned}
J_{\delta}\left(z\left(t_0\right)\right) &= \frac{\delta}{2}\|\mathcal{A}^{\frac{1}{2}} z\left(t_0\right)\|^2-\frac{1}{q+1}\|z\left(t_0\right)\|_{q+1}^{q+1} \\
&= \delta\left(\frac{1}{2}-\frac{1}{q+1}\right)\|\mathcal{A}^{\frac{1}{2}} z\left(t_0\right)\|^2+\frac{1}{q+1}\left(\delta\|\mathcal{A}^{\frac{1}{2}} z\left(t_0\right)\|^2-\|z\left(t_0\right)\|_{q+1}^{q+1}\right) \\
&\geq \frac{(q-1)\,\delta}{2\,(q+1)}\|\mathcal{A}^{\frac{1}{2}} z\left(t_0\right)\|^2 \\
&= d(\delta)\,.
\end{aligned}
$$

This contradicts

$$E(0)=E(t)=\frac{1}{2}\|z_t(t)\|^2+J_1(z(t))<d(\delta)\,,\ \forall t \in[0, T)\,.$$

Hence $z(t) \in W_{\delta}$ *for all* $t \in[0, T)$.

ii) Arguing by contradiction, suppose that $t_0 \in(0, T)$ *is the first time such that* $z\left(t_0\right) \in \partial V_{\delta}$. *The remainder of proof is the same as that in (i), and so it is omitted here.*

3 Problem (1) with the subcritical initial energy $(E(0)<d(\delta))$

In this section, we proved the global existence and blow up of solutions for problem (1) with the subcritical initial energy.

3.1 Global existence when $0<E(0)<d(\delta)$

In this part, we proved the global existence of solutions for problem (1).

Theorem 3.1 *Suppose that* $z_0 \in W_{\delta},\ z_1 \in L^2(\Omega)$ *and* $0<E(0)<d(\delta)$. *Then problem (1) admits a solution* $z(t) \in W_{\delta}$ *for all* $t \in[0, \infty)$.

Proof 3.2 *Let* $\left\{w_j\right\}_{j=1}^{\infty}$ *be a completed orthogonal basis of* $H_0^m(\Omega)$ *and an orthonormal basis of* $L^2(\Omega)$. *We construct the approximate solution*

$$z_n(t)=\sum_{j=1}^n \xi_{jn}(t)\, w_j, \quad n=1,2,3, \ldots,$$

solving the problem

$$\left(z_{ntt}(t), w_j\right)+\left(\mathcal{A}^{\frac{1}{2}} z_n(t), \mathcal{A}^{\frac{1}{2}} w_j\right)=\left(\left|z_n(t)\right|^{q-1} z_n(t), w_j\right), \quad j=1,2,3, \ldots ., n, \tag{5}$$

$$z_n(0) = \sum_{j=1}^{n} \xi_{jn}(0) w_j \to z_0 \in H_0^m(\Omega), \tag{6}$$

$$z_{nt}(0) = \sum_{j=1}^{n} \xi'_{jn}(0) w_j \to z_1 \in L^2(\Omega). \tag{7}$$

Multiplying eq. (5) by $\xi'_{jn}(t)$ *and summing for* j, *we obtain*

$$\frac{d}{dt}\left(\frac{1}{2}\|z_{nt}(t)\|^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} z_n(t)\|^2 - \frac{1}{q+1}\|z_n(t)\|_{q+1}^{q+1}\right) = 0. \tag{8}$$

Integrating (8) with respect to t *on* $[0,t]$, *we arrive at*

$$E_n(t) = \frac{1}{2}\|z_{nt}(t)\|^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} z_n(t)\|^2 - \frac{1}{q+1}\|z_n(t)\|_{q+1}^{q+1} = E_n(0) \tag{9}$$

where

$$E_n(0) = \frac{1}{2}\|z_{nt}(0)\|^2 + \frac{1}{2}\|\mathcal{A}^{\frac{1}{2}} z_n(0)\|^2 - \frac{1}{q+1}\|z_n(0)\|_{q+1}^{q+1}.$$

Recalling (6) and (7) yields $E_n(0) \to E(0)$, $0 < E_n(0) < d(\delta)$ *and* $z_n(0) \in W_\delta$ *for sufficiently large* n. *By similar arguments in (i) of Theorem 3, we get* $z_n(0) \in W_\delta$ *for all* $t \in [0,\infty)$. *As a result*

$$\|\mathcal{A}^{\frac{1}{2}} z_n(t)\| < \left[\frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}\right]^{\frac{1}{2}}, \forall t \in [0,\infty)$$

and

$$\|z_n(t)\|_{q+1} \le c\|\mathcal{A}^{\frac{1}{2}} z_n(t)\| < c\left[\frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}\right]^{\frac{1}{2}}, \ \forall t \in [0,\infty).$$

When $\|\mathcal{A}^{\frac{1}{2}} z_n(t)\| \neq 0$, *in terms of (i) in Lemma 1 and (9), we get*

$$\|z_n(t)\|^2 < 2d(\delta), \ \forall t \in [0,\infty).$$

When $\|\mathcal{A}^{\frac{1}{2}} z_n(t)\| = 0$, *by means of (9), the above inequality remains valid.*

Therefore, there exist z, $\mathcal{X}$ *and a subsequence of* $\{z_n\}$, *always relabeled as the same and we shall not repeat, such that, as* $n \to \infty$,

$$z_n \rightharpoonup z \text{ weakly star in } L^\infty(0,\infty; H_0^m(\Omega)), \text{ and } z_n \to z \text{ a.e. in } \Omega \times [0,\infty),$$

$$z_{nt} \rightharpoonup z_t \text{ weakly star in } L^\infty\left(0,\infty; L^2(\Omega)\right),$$

$$|z_n|^{q-1} z_n \rightharpoonup \mathcal{X} \text{ weakly star in } L^\infty(0,\infty; L^r(\Omega)), \ r = \frac{q+1}{q}.$$

According to [5] (Chapter 1, Lemma 1.3), we get $\mathcal{X} = |z|^{q-1}z$.

For fixed j*, integrating (5) with respect to* t *and taking* $n \to \infty$*, we obtain*

$$(z_t(t), w_j) + \int_0^t \left(\mathcal{A}^{\frac{1}{2}} z(\tau), \mathcal{A}^{\frac{1}{2}} w_j\right) d\tau = \int_0^t \left(|z(\tau)|^{q-1} z(\tau), w_j\right) d\tau + (z_1, w_j).$$

Also, it is easy to see from (6) and (7) that $z(0) = z_0$ *in* $H_0^m(\Omega)$, $z_t(0) = z_1$ *in* $L^2(\Omega)$. *Consequently,* z *is a solution of problem (1) in the sense of Definition 2. In addition, according to (i) in Theorem 3, we get* $z(t) \in W_\delta$ *for all* $t \in [0, \infty)$.

3.2 Blow up when $E(0) < d(\delta)$

In this part, we proved the blow up of solutions for problem (1).

Theorem 3.3 *Suppose that* $z_0 \in V_\delta$, $z_1 \in L^2(\Omega)$ *and* $E(0) < d(\delta)$. *Then solutions of problem (1) blow up in finite time.*

Proof 3.4 *Let* z *be a solution of problem (1). Now, we prove* $T < \infty$*. If it is not true, then* $T = \infty$*. We define the auxiliary function*

$$K(t) = \|z\|^2, \ t \in [0, \infty).$$

Then by taking the derivative of $K(t)$, *we obtain*

$$K'(t) = 2\int z z_t dx,$$

and

$$\begin{aligned} K''(t) &= 2\int \left[z_t^2 + z z_{tt}\right] dx \\ &= 2\|z_t\|^2 + 2\|z\|_{q+1}^{q+1} - 2\|\mathcal{A}^{\frac{1}{2}} z\|^2 \\ &= (q+3)\|z_t\|^2 + (q-1)\|\mathcal{A}^{\frac{1}{2}} z\|^2 - 2(q+1)E(0) \qquad (10) \end{aligned}$$

When $0 < E(0) < d(\delta)$, *by virtue of* $z_0 \in V_\delta$ *and (ii) in theorem 3, we get* $z(t) \in V_\delta$ *for all* $t \in [0, \infty)$ *and so*

$$\|\mathcal{A}^{\frac{1}{2}} z\|^2 > \frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}.$$

Hence

$$\begin{aligned} (q-1)\delta\|\mathcal{A}^{\frac{1}{2}} z\|^2 &> 2(q+1)d(\delta) \\ &> 2(q+1)E(0), \end{aligned}$$

which together with (10), gives

$$K''(t) > (q+3)\|z_t\|^2.$$

When $E(0) \le 0$*, on account of (10), the above inequality still holds. Therefore, there exists a* $t_0 > 0$ *such that* $K(t_0) > 0$ *and* $K'(t) \ge K'(t_0) > 0$ *for a.e.* $t \in [t_0, \infty)$*. Then*

$$K(t) \ge K'(t_0)(t-t_0) + K(t_0) > 0, \quad t \in [t_0, \infty).$$

Thans to Cauchy-Schwarz inequality, we have

$$K(t)K''(t) - \frac{(q+3)}{4}[K'(t)]^2 \ge (q+3)\left[\|z\|^2\|z_t\|^2 - (z, z_t)^2\right] \ge 0.$$

Thus

$$\left[K^{-\beta}(t)\right]' = -\beta K^{-(1+\beta)}(t)K'(t) < 0,$$

and

$$\left[K^{-\beta}(t)\right]'' = \frac{-\beta}{K^{\beta+2}(t)}\left[K(t)K''(t) - (\beta+1)[K'(t)]^2\right] \le 0,$$

for a.e. $t \in [t_0, \infty)$ *, where* $\beta = \frac{q-1}{4}$*. Then there exists a* T_0 *such that*

$$\lim_{t \to T_0} K(t) = \infty,$$

which conradicts $T = \infty$*. Thus, the proof is complete.*

4 Problem (1) with the critical energy ($E(0) = d(\delta)$)

In this section, we proved the global existence and blow up of solutions for problem (1) with the critical initial energy.

4.1 Global existence when $E(0) = d(\delta)$

Lemma 4.1 *Suppose that* $z \in H_0^m(\Omega)$ *and* $\|\mathcal{A}^{\frac{1}{2}}z\| \ne 0$*.* $J_\delta(\rho z)$ *is strictly increasing for* $\rho \in (0, \rho_{*,\delta})$*, strictly decreasing for* $\rho \in (\rho_{*,\delta}, \infty)$*, and attains the maximum at* $\rho = \rho_{*,\delta}$*.*

Proof 4.2 *By the definition of* $J_\delta(z)$*, we get*

$$\begin{aligned} J_\delta(\rho z) &= \frac{\delta}{2}\|\mathcal{A}^{\frac{1}{2}}(\rho z)\|^2 - \frac{1}{q+1}\|\rho z\|_{q+1}^{q+1} \\ &= \frac{\delta}{2}\rho^2\|\mathcal{A}^{\frac{1}{2}}z\|^2 - \frac{\rho^{q+1}}{q+1}\|z\|_{q+1}^{q+1}. \end{aligned}$$

Thus
$$\frac{d}{d\rho}\left(J_\delta\left(\rho z\right)\right)=\delta\rho\|\mathcal{A}^{\frac{1}{2}}z\|^2-\rho^q\|z\|_{q+1}^{q+1}.$$
Clearly, there is a $\rho_{*,\delta}=\rho_{*,\delta}\left(z\right)>0$ *such that*
$$\delta\|\mathcal{A}^{\frac{1}{2}}z\|^2=\rho_{*,\delta}^{q-1}\|z\|_{q+1}^{q+1},$$
i.e.,
$$\frac{d}{d\rho}\left(J_\delta\left(\rho z\right)\right)|_{\rho_{*,\delta}}=0.$$
Moreover, $\frac{d}{d\rho}\left(J_\delta\left(\rho z\right)\right)>0$ *for* $\rho\in\left(0,\rho_{*,\delta}\right)$*,* $\frac{d}{d\rho}\left(J_\delta\left(\rho z\right)\right)<0$ *for* $\rho\in\left(\rho_{*,\delta},\infty\right)$*.*

Theorem 4.3 *Assume that* $z_0\in W_\delta,\ z_1\in L^2\left(\Omega\right)$ *and* $E\left(0\right)=d\left(\delta\right)$*. Then problem (1) admits* $z\left(t\right)\in\overline{W_\delta}=W_\delta\cup\partial W_\delta$ *for all* $t\in[0,\infty)$*.*

Proof 4.4 *We will prove it in two cases.*

i) $\|\mathcal{A}^{\frac{1}{2}}z_0\|^2\neq0$*.*

Let $z_{0k}=\lambda_k z_0$*,here* $\lambda_k=1-\frac{1}{k},\ \ k=2,3,....$ *Now, we consider the following problem*
$$\begin{cases} z_{tt}+\mathcal{A}z=|z|^{q-1}z, & (x,t)\in\Omega\times(0,\infty)\\ \frac{\partial^i z(x,t)}{\partial v^i}=0,\ i=0,1,...,m-1, & (x,t)\in\partial\Omega\times[0,\infty)\\ z\left(x,0\right)=z_{0k}\left(x\right),\ z_t\left(x,0\right)=z_1\left(x\right), & x\in\Omega\end{cases}\tag{11}$$
whose energy functional is
$$E_{kt}\left(t\right)=\frac{1}{2}\|z_{kt}\|^2+\frac{1}{2}\|\mathcal{A}^{\frac{1}{2}}z_k\|^2-\frac{1}{q+1}\|z_k\|_{q+1}^{q+1}.$$

From $z_0\in W_\delta$ *and Lemma 1 it follows that*
$$\delta\|\mathcal{A}^{\frac{1}{2}}z_0\|^2>\|z_0\|_{q+1}^{q+1},\tag{12}$$
Thus
$$\delta\|\mathcal{A}^{\frac{1}{2}}z_0\|^2>\lambda_k^{q-1}\|z_0\|_{q+1}^{q+1},$$
and so
$$\delta\|\mathcal{A}^{\frac{1}{2}}z_{0k}\|^2>\|z_{0k}\|_{q+1}^{q+1}.$$
Which implies that
$$J_\delta\left(z_{0k}\right)=\frac{\delta}{2}\|\mathcal{A}^{\frac{1}{2}}\left(z_{0k}\right)\|^2-\frac{1}{q+1}\|z_{0k}\|_{q+1}^{q+1}>0.$$
It follows from (12) and the proof of Lemma 6 that there exists a $\rho_{*,\delta}=\rho_{*,\delta}(z_0)>1$ *such that* $J_\delta\left(\rho z_0\right)$ *attains its maximum. Thus, according to Lemma*

6, $J_\delta(\rho z_0)$ is strictly increasing on $[\lambda_k, 1]$, and $J_1(\lambda_k z_0) < J_1(z_0)$. Consequently,

$$\begin{aligned} 0 &< E_k(0) \\ &= \frac{1}{2}\|z_1\|^2 + J_1(z_{0k}) \\ &< \frac{1}{2}\|z_1\|^2 + J_1(z_0) \\ &= E(0) \\ &= d(\delta). \end{aligned}$$

In terms of Theorem 4, for each k problem (11) admits a solution $z_k(t) \in W_\delta$ for all $t \in [0,\infty)$ satisfying

$$(z_{kt}(t), v) + \int_0^t \left(\mathcal{A}^{\frac{1}{2}} z_k(\tau), \mathcal{A}^{\frac{1}{2}} v\right) d\tau = \int_0^t \left(|z_k(\tau)|^{q-1} z_k(\tau), v\right) d\tau + (z_1, v), \tag{13}$$

for all $v \in H_0^m(\Omega)$. Consequently,

$$\|\mathcal{A}^{\frac{1}{2}} z_k(t)\| < \left[\frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}\right]^{\frac{1}{2}}, \quad \forall t \in [0,\infty). \tag{14}$$

When $\|\mathcal{A}^{\frac{1}{2}} z_k(t)\| \neq 0$, in terms (i) of Lemma 1 $E_k(t) = E_k(0) < d(\delta)$, we get

$$\|z_{kt}(t)\|^2 < 2d(\delta), \quad \forall t \in [0,\infty).$$

When $\|\mathcal{A}^{\frac{1}{2}} z_k(t)\| = 0$, the above inequality still holds. By the compactness arguments used by the proof of Theorem 6, there exists a z such that, as $k \to \infty$ in (13),

$$(z_t(t), v) + \int_0^t \left(\mathcal{A}^{\frac{1}{2}} z(\tau), \mathcal{A}^{\frac{1}{2}} v\right) d\tau = \int_0^t \left(|z(\tau)|^{q-1} z(\tau), v\right) d\tau + (z_1, v).$$

Moreover, it follows from (11) that $z(0) = z_0$ in $H_0^m(\Omega)$, $z_t(0) = z_1$ in $L^2(\Omega)$. Therefore, z is a solution of problem (1). By virtue of (14), we obtain

$$\|\mathcal{A}^{\frac{1}{2}} z(t)\| \le \lim_{m\to\infty} \inf \|\mathcal{A}^{\frac{1}{2}} z_k(t)\| \le \left[\frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}\right]^{\frac{1}{2}}, \quad \forall t \in [0,\infty).$$

Hence $z(t) \in \overline{W_\delta}$ for all $t \in [0,\infty)$.

ii) $\|\mathcal{A}^{\frac{1}{2}} z(0)\| = 0$.

Let $z_{1k} = \lambda_k z_1$, $\lambda_k = 1 - \frac{1}{k}$, $k = 2, 3, \ldots$. Next, we consider

$$\begin{cases} z_{tt} + \mathcal{A}z = |z|^{q-1} z, & (x,t) \in \Omega x (0,\infty), \\ \frac{\partial^i z(x,t)}{\partial v^i} = 0, \ i = 0, 1, \ldots, m-1, & (x,t) \in \partial\Omega x [0,\infty), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_{1k}(x), & x \in \Omega. \end{cases} \tag{15}$$

In this case, due to the fact that $J_1(z_0) = 0$ *and* $\frac{1}{2}\|z_1\|^2 = E(0)$, *we obtain*

$$0 < E_k(0) = \frac{1}{2}\|z_{1k}\|^2 + J_1(z_0) = \frac{1}{2}\|\lambda_k z_1\|^2 < E(0) = d(\delta).$$

Thus it follows from Theorem 6 that, for each k, *problem (15) admits a solution* $z_k(t) \in W_\delta$ *for all* $t \in [0, \infty)$. *The remainder of proof is similar to that in (i). The proof of is complete.*

Theorem 4.5 *Suppose that* z *be a solution of problem (1) on* $\Omega \times [0, T)$. *Suppose further that* $z_0 \in V_\delta$, $z_1 \in L^2(\Omega)$, $E(0) = d(\delta)$ *and* $(z_0, z_1) \geq 0$. *Then* $z(t) \in V_\delta$ *for all* $t \in [0, T)$.

Proof 4.6 *Assume that* $z(t) \notin V_\delta$ *for some* $0 < t < T$. *Then we see from* $z_0 \in V_\delta$ *that there exists the first time* $0 < t_0 < T$ *such that* $z(t_0) \in \partial V_\delta$. *Hence*

$$\|\mathcal{A}^{\frac{1}{2}} z(t_0)\| = \left[\frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}\right]^{\frac{1}{2}} \tag{16}$$

and

$$\|\mathcal{A}^{\frac{1}{2}} z(t)\| > \left[\frac{2(q+1)}{q-1}\frac{d(\delta)}{\delta}\right]^{\frac{1}{2}}, \ \forall t \in [0, t_0). \tag{17}$$

By (16) and (ii) in Lemma 1, we obtain

$$\delta\|\mathcal{A}^{\frac{1}{2}} z(t_0)\|^2 \geq \|z(t_0)\|_{q+1}^{q+1}$$

Hence

$$\begin{aligned} J_\delta(z(t_0)) &= \frac{\delta}{2}\|\mathcal{A}^{\frac{1}{2}} z(t_0)\|^2 - \frac{1}{q+1}\|z(t_0)\|_{q+1}^{q+1} \\ &\geq \frac{(q-1)\delta}{2(q+1)}\|\mathcal{A}^{\frac{1}{2}} z(t_0)\|^2 \\ &= d(\delta). \end{aligned} \tag{18}$$

Set

$$K(t) = \|z\|^2, \ t \in [0, T).$$

A direct calculation yields

$$K'(t) = 2(z(t), z_t(t)).$$

From $E(0) = d(\delta)$, *(10) and (17), it follows that* $K''(t) > 0$ *for* $t \in [0, t_0)$. *Hence* $K'(t)$ *is increasing on* $[0, t_0]$. *We further get* $K'(t_0) > K'(0)$, *which, together with* $K'(0) = (z_0, z_1) \geq 0$, *gives* $K'(t_0) = (z(t_0), z_t(t_0)) > 0$. *Thus*

$$\|z(t_0)\|.\|z_t(t_0)\| \geq ((z(t_0), z_t(t_0))) > 0,$$

$$E(t_0) = \frac{1}{2}\|z_t(t_0)\|^2 + J_1(z(t_0)) = E(0) = d(\delta),$$

we get $J_1(z(t_0)) < d(\delta)$, *which contradicts (18). Therefore,* $z(t) \in V_\delta$ *for all* $t \in [0, T)$.

4.2 Blow up when $E(0) = d(\delta)$

Theorem 4.7 *Suppose that $z_0 \in V_\delta,\ z_1 \in L^2(\Omega)\,,\ E(0) = d(\delta)$ and $(z_0, z_1) \geq 0$. Then the solutions of problem (1) blow up in finite time.*

Proof 4.8 *Let z be a solution of problem (1). Next, we prove $T < \infty$. If it is not true, then $T = \infty$. From $z_0 \in V_\delta,\ (z_0, z_1) \geq 0$ and Theorem 8, it follows that $z(t) \in V_\delta$ for all $t \in [0, \infty)$. Hence*

$$\|\mathcal{A}^{\frac{1}{2}} z(t)\|^2 > \frac{2(q+1)}{q-1} \frac{d(\delta)}{\delta}.$$

Combining this with $E(0) = d(\delta)\,,$ we have

$$\begin{aligned} \delta (q-1) \|\mathcal{A}^{\frac{1}{2}} z(t)\|^2 &> 2(q+1) d(\delta) \\ &= 2(q+1) E(0)\,. \end{aligned}$$

Hence, for $K(t)$ introduced in the proof of Theorem 8, it follows from (10) that

$$K''(t) > (q+3) \|z_t(t)\|^2,$$

for a.e. $t \in [0, \infty)\,.$ The remainder of proof is the same as that in Theorem 5, and so it is omitted here.

5 Open problems

In the present work we proved the global existence and blow up of solutions for problem (1) using the potential well method under subcritical $(E(0) < d(\delta))$ and critical $(E(0) = d(\delta))$ energy. The asymptotic behavior of the problem can be studied. Additionally, the global existence can be examined for $E(0) > d(\delta)\,.$

References

[1] J. A. Esquivel-Avila, Qualitative analysis of a nonlinear wave equation, Discrete Contin. Dyn. Syst., 10 (2004) 787-804.

[2] F. Gazzola, M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, Ann. Inst. H. Poincar e Anal. Non Lin eaire, 23 (2006) 185-207.

[3] R. Ikehata, Some remarks on the wave equations with nonlinear damping and source terms, Nonlinear Anal., 27 (1996) 1165-1175.

[4] Y. Liu, W. Li, A family of potential wells for a wave equation, Electronic Research Archive, 28(2) (2020) 807-820.

[5] J. L. Lions, Quelques Methodes de Resolution des Problemes aux Limites non Lineaires, Dunod, Paris, 1969.

[6] L.E. Payne, D.H.Sattinger, Sadle points and instability of nonlinear hyperbolic equations, Israel J. Math., 22 (1975) 273-303.

[7] D.H. Sattinger, On global solution of nonlinear hyperbolic equations, Arch. Rational Mech. Anal., 30 (1968) 148-172.

[8] M. Shahrouzi, Asymptotic behavior of solutions for a nonlinear viscoelastic higher-order p(x)-Laplacian equation with variable-exponent logarithmic source term, Bol. Soc. Mat. Mex., 29 (2023) 1-20.

[9] G. Todorova, Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms, J. Math. Anal. Appl., 239 (1999) 213-226.

[10] R. Xu, Y. Yang, S. Chen, J. Shen, T. Yu, Z. Xu, Global well-posedness of semilinear hyperbolic equations, parabolic equations and Schrödinger equations, Electron, J. Differantial Equations, 2018 (2018) 1-52.

[11] R. Xu, M. Zhang, S. Chen, Y. Yang and J. Shen, The initial-boundary value problems for a class of six order nonlinear wave equation, Discrete Contin. Dyn. Syst., 37 (2017) 5631-5649.

[12] E. Vitillaro, A potential well theory for the wave equation with nonlinear source and boundary damping terms, Glasgow Mathematical Journal, 44(3) (2002) 375-395.

[13] S. Wang, H. Xue, Global solution for a generalized Boussinesq equation, Appl. Math. Comput., 204 (2008) 130-136.

[14] L. Yacheng, On potential wells and vacuum isolating of solutions for semilinear wave equations. Journal of Differential Equations, 192(1) (2003) 155-169.