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Survey on $m-$subharmonic functions and m-positive currents

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Abstract

In this survey we gather several results related to the set of m-subharmonic functions and m-positive currents. We focus on recent studies of the Hessian operator acting on the set of m-subharmonic functions as well as the same operator when it is associated to an m-positive closed current.

Keywords: *m-subharmonic function, currents, Capacity, Hessian operator.*

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1 Introduction

In this survey we fix Ω to be an open subset of $\mathbb{C}^n$ and m an integer such that $1 \leq m \leq n$. The notion of plurisubharmonic functions (psh) represents multidimensional generalizations of subharmonic functions (sh). It is therefore not surprising that these two classes of functions share many similar properties. However there are several properties that make a psh function different from an sh function, namely the property of symmetry: the notion of plurisubharmonicity is stable by change in holomorphic coordinates which is not the case for sh functions on $\mathbb{C}^n$ for $n > 1$. Also the Liouville type property shows that there is a difference between the two notions already cited. Indeed it is known that a psh function is bounded on $\mathbb{C}^n$ if and only if it is constant and this is not the case for sh functions in higher dimensions. The notion of m-subharmonic function (see [4], [28, 27]) interpolates between subharmonicity and plurisubharmonicity. The corresponding nonlinear potential theory is therefore expected to share the joint properties of the potential and the pluripotential theory. Indeed in the works of Li [26], Blocki [4], Chinh [28], Dhouib and Elkhadhra [13], Dinew and Kolodzeij [14] and many others the m-subharmonic potential theory (m-sh) has been fully developed. For every locally bounded m-sh function u, Li [26] and Blocki [4] defined the Hessian operator of u as

$$H_m(u) := (dd^c u)^m \wedge (dd^c|z|^2)^{n-m}.$$

The operator H_m generlizes the Monge-Ampère operator when $m = n$ and in this case the operator was extensivly studied by Bedford and Taylor [1], Demailly [12] and Cegrell [5, 6].
For a given non negative measure μ defined on Ω one can study the following Hessian equation on the set of $m$sh functions:

$$H_m(.) = \mu \qquad (E_1).$$

The Hessian equation coincides in the case $m = 1$ with the Poisson equation and with the Monge-Ampère equation when $m = n$. Unlike complex Monge-Ampère equations, where the eigenvalues of the associated operator are positive, complex Hessian equations are more dicult to manipulate. The m-subharmonic functions do not have "nice" mean value properties. They are not invariant by holomorphic maps.
Real Hessian equations have been studied intensively in recent years with numerous applications (see [29] and its references). Blocki [4] developed the first elements of a local potential theory for the Hessian equation on an open set of $\mathbb{C}^n$ analogous to Bedford and Taylor [1]. This survey is organized as follows: In the first part we recall basic properties of several classes of m-subharmonic functions as well as necessary elements of pluripotential theory. In the second part we summarize different results on solving the equation (E) when it is associated to a weight χ. So we present several recent results in [19, 33, 20] where we studied a general version of the complex Hessian equation on an m-hyperconvex domain of $\mathbb{C}^n$. This equation was the objective of different studies not only because it represents an example of a partial differential equation but also because it has many applications in several geometric problems. It consists of looking for a solution, in the set of functions m-sh, of the following equation

$$-\chi(.)H_m(.) = \mu \qquad (E)$$

where μ is a measure defined on Ω and χ a negative increasing function defined on $\mathbb{R}_-$. The objective was then to find sufficient conditions on μ and χ ensuring the existence of a solution for the equation (E).

The search for such a solution must obviously be done in the domain of definition of the operator $\chi(.)H_m(.)$. So it is trivial to involve the complex energy classes defined by Benelkourchi, Guedj and Zeriahi [3] and generalized by V.V. Hung [22], namely the classes $\mathcal{E}_{m,\chi}(\Omega)$ and $\mathcal{E}_\chi(\Omega)$.
In the special case $\chi \equiv -1$, the equation (E) is exactly the equation (E_1) which was solved by Lu [28] under the assumption that the measure μ has no mass on all m-polar sets. This problem was improved by V.V. Hung and N. V. Phu [23] by demonstrating that if μ is a finite Radon measure and if (E_1) has a subsolution then there exists a solution u to the equation (E_1) that belongs to $\mathcal{E}_m(\Omega)$.
In the case where χ is not identically equal to -1 but $m = n$, Czyz [10] has proven the existence of a solution to the equation (E) belonging to the class $\mathcal{E}_\chi(\Omega)$. Here we treat the general case when $m \neq n$ and χ not identically equal to -1.
Finally we will study a more general case when we associate to every m-positive current its Hessian equation and we give sufficient condition to ensure the existence of its solution. The main results of this section may be found in recent work of [17, 18]. We recall first the classes analogous to those of Cegrell $\mathcal{E}^T_{0,m}(\Omega)$ and $\mathcal{E}^T_{p,m}(\Omega)$ which generalize the classes given by Lu [28]. Using techniques similar to those of Cegrell [5, 6] we proved in [17] that the operator $(dd^c.)^q \wedge T$ is well defined on the class $\mathcal{E}^T_{p,m}(\Omega)$. As the operator is now well defined we moved on to treating its properties all while drawing inspiration from the properties that the classical Monge-Ampère operator had. In this direction we cite several recent result related to the problem of quasicontinuity of any function $u \in \mathcal{E}^T_{p,m}(\Omega)$. We then prove that any function $u \in \mathcal{E}^T_{p,m}(\Omega)$ is $Cap_{m,T}-$quasicontinuous.
The end of this section will be devoted to study m-positive currents in the sense of Dhouib and Elkhadhra [13] where we discuss some theorems of convergenceof the Hessian measures in this case.

2 Preliminaries

This section is devoted to recall some basic properties of $m-$subharmonic functions that represent admissible functions for the complex Hessian equation. Throughout this paper we denote by $d := \partial + \overline{\partial}$ and $d^c := i(\overline{\partial} - \partial)$. The standard Kähler form defined on $\mathbb{C}^n$ will be denoted as $\beta := dd^c|z|^2$ and we will denote by $C_{(1,1)}$ the set of all forms of bidegres $(1,1)$ with constant coefficients. Set

$$\widehat{\Gamma}_m := \{\alpha \in C_{(1,1)}; \alpha \wedge \beta^{n-j} \geq 0; \forall j \in \{1, ..., m\}\}.$$

Definition 2.1 *Let $f : \Omega \to \mathbb{R} \cup \{-\infty\}$ be a subharmonic function. The* *function f is called m-subharmonic if for all $\zeta_1, \cdots, \zeta_{m-1} \in \widehat{\Gamma}_m(\Omega)$ one has*

$$dd^c f \wedge \zeta_1 \wedge \cdots \wedge \zeta_{m-1} \geq 0.$$

We denote by $\mathcal{SH}_m(\Omega)$ the cone of $m-$subharmonic functions defined on Ω. We cite below basic properties of m-subharmonic functions where its proofs can be found in [4], [24],[28] and [27].

Remarks 2.2 *1. The set $\mathcal{SH}_n(\Omega)$ coincides with the set of plurisubharmonic functions on Ω.*

2. *If $u, v \in SH_m(\Omega)$ then $\lambda u + \mu v \in SH_m(\Omega), \forall \lambda, \mu > 0$.*

3. *$PSH(\Omega) = SH_n(\Omega) \subset \cdots \subset SH_m(\Omega) \subset \cdots \subset SH_1(\Omega) = SH(\Omega)$.*

4. *If u is m-subharmonic on Ω then the standard regularizations $u * \chi_\epsilon$ are also $m-$subharmonic on $\Omega_\epsilon := \{x \in \Omega \ / \ d(x, \partial\Omega) > \epsilon\}$.*

5. *If $(u_i)_j$ is a decreasing sequence of $m-$subharmonic functions then $u := \lim u_j$ is either $m-$subharmonic or identically equal to $-\infty$.*

One can construct an example of m-sh subharmonic which is not $(m+1)$-sh. It suffices to consider the function

$f(z) := 2|z_1|^2 + 2|z_2|^2 - |z_3|^2$ for $z \in \mathbb{C}^3$. Indeed it is easy to check that $f \in \mathcal{SH}_2(\mathbb{C}^3) \setminus \mathcal{SH}_3(\mathbb{C}^3)$.

In the case of locally bounded m-sh functions, one can define, by induction, a closed nonnegative current in the same manner as Bedford and Taylor [1] have defined it for plurisubharmonic functions

$$dd^c f_1 \wedge \ldots \wedge dd^c f_k \wedge \beta^{n-m} := dd^c(f_1 dd^c f_2 \wedge \ldots \wedge dd^c f_k \wedge \beta^{n-m}),$$

where $f_1, \ldots, f_k \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$. In particular, for a given $m-$sh function $f \in \mathcal{SH}_m(\Omega) \cap L^\infty_{loc}(\Omega)$, we define the nonnegative Hessian measure of f as follows

$$H_m(f) = (dd^c f)^m \wedge \beta^{n-m}.$$

2.1 Cegrell classes of m-sh functions and $m-$capacity

Definition 2.3 *1. A bounded domain Ω in $\mathbb{C}^n$ is said to be m-hyperconvex if the following property holds for some continuous m-sh function $\rho : \Omega \to \mathbb{R}^-$:*
$\{\rho < c\}$ is relatively compact in Ω for every $c < 0$.

2. *A set $M \subset \Omega$ is called $m-$polar if there exists $u \in \mathcal{SH}_m(\Omega)$ such that*

$$M \subset \{u = -\infty\}.$$

Throughout the rest of this survey, we denote by Ω an m-hyperconvex domain of $\mathbb{C}^n$. In [5] and [6], Cegrell introduced and studied new fundamental classes of negative psh functions to extend the domain of definition of the Monge-ampère operator. Such classes are very useful in the resolution of the Dirichlet problem. To generalize the above classes, Lu [28, 27] introduced the following classes of m-sh functions that coincides in the case $m = n$ with the Cegrell's one. We recall below the definitions of those classes.

Definition 2.4 *We denote by:*

$$\mathcal{E}_m^0(\Omega) := \mathcal{E}_m^0 = \{f \in \mathcal{SH}_m^-(\Omega) \cap L^\infty(\Omega);\ \lim_{z\to\xi} f(z) = 0\ \forall \xi \in \partial\Omega\ ,\ \int_\Omega H_m(f) < +\infty\},$$

$$\mathcal{F}_m(\Omega) = \mathcal{F}_m = \{f \in \mathcal{SH}_m^-(\Omega);\ \exists (f_j) \subset \mathcal{E}_m^0,\ f_j \searrow f \text{ in } \Omega\ \sup_j \int_\Omega H_m(f_j) < +\infty\},$$

and

$$\mathcal{E}_m(\Omega) := \mathcal{E}_m = \{f \in \mathcal{SH}_m^-(\Omega) : \forall U \Subset \Omega, \exists\ f_U \in \mathcal{F}_m(\Omega);\ f_U = f \text{ on } U\}.$$

2.2 Energy Complex classes

Definition 2.5 *A function $f \in SH_m(\Omega)$ is said to be m-maximal if for every $g \in SH_m(\Omega)$ such that $g \le f$ outside some comapct of Ω one has that $g \le f$ in Ω.*

The previous notion represents an essential tool in the study of the Hessian operator since Blocki [4] proved that every m-maximal fucntion $f \in \mathcal{E}_m(\Omega)$ satisfies $H_m(f) = 0$ and consequently it is a solution of the homogeneous complex Hessian equation. We then recall the construction of the m-maximal functions.

Let $(\Omega_j)_j$ be a sequence of m-pseudoconvexes subsets of Ω such that $\Omega_j \Subset \Omega_{j+1}$, $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$ and for every j there exists a smooth strictly m-sh function φ in a neighborhood V of Ω_j such that $\Omega_j := \{z \in V / \varphi(z) < 0\}$.

Definition 2.6 *Let $f \in SH_m^-(\Omega)$ and $(\Omega_j)_j$ the sequence defined above. Consider f^j the function defined by:*

$$f^j = \sup\left\{\psi \in SH_m(\Omega) :\ \psi_{|\Omega\setminus\Omega_j} \le f\right\} \in SH_m(\Omega),$$

and denote by $\widetilde{f} := (\lim_{j\to+\infty} f^j)^*$, the function $\widetilde{f}$ is the smallest m-maximal m-sh majorant of f.

If $f \in \mathcal{E}_m(\Omega)$ then using [27] and [4] $\widetilde{f} \in \mathcal{E}_m(\Omega)$ and is m-maximale on Ω. We will denote by $\mathcal{MSH}_m(\Omega)$ the set of all m-maximal functions in Ω.
We cite below some fundamental properties of $\mathcal{MSH}_m(\Omega)$.

Proposition 2.7 *[4] If $f, g \in \mathcal{E}_m(\Omega)$ and $\alpha \in \mathbb{R}$, $\alpha \geq 0$, then*

1. $\widetilde{f+g} \geq \widetilde{f} + \widetilde{g}$.
2. $\widetilde{\alpha f} = \alpha \widetilde{f}$.
3. *If* $f \leq g$ *then* $\widetilde{f} \leq \widetilde{g}$.
4. $\mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega) = \{f \in \mathcal{E}_m : \ \widetilde{f} = f\}$.

In [22], author introduce the class $\mathcal{N}_m(\Omega) := \{f \in \mathcal{E}_m : \ \widetilde{f} = 0\}$ which is equal, when $m = n$, to the Cegrell class $\mathcal{N}(\Omega)$ defined and studied in [6]. It is easy to check that $\mathcal{N}_m(\Omega)$ is a convex cone and that the following inclusions hold

$$\mathcal{E}_m^0(\Omega) \subset \mathcal{F}_m(\Omega) \subset \mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega).$$

Definition 2.8 *Let $\mathcal{L}_m \in \{\mathcal{E}_m^0, \mathcal{F}_m, \mathcal{N}_m\}$ and $u \in \mathcal{E}_m(\Omega)$. A function $f \in SH_m(\Omega)$ belongs $\mathcal{L}_m(\Omega, u)$ ($\mathcal{L}_m(u)$) if there exists a function $\psi \in \mathcal{L}_m$ such that $u \geq f \geq \psi + u$.*

In the case when $u = 0$, we get $\mathcal{L}_m(u) = \mathcal{L}_m$. For $u \in \mathcal{E}_m(\Omega)$, one can defined the classes

$$\mathcal{N}_m^a(\Omega) := \{f \in \mathcal{N}_m : \ H_m(f)(P) = 0, \ \forall P \ m-polar\}\},$$

and

$$\mathcal{N}_m^a(\Omega, u) := \{f \in \mathcal{E}_m \ / \ \exists \psi \in \mathcal{N}_m^a \ \ tel \ que \ \ u \geq f \geq \psi + u\}.$$

3 Main results: Hessian equations acting on weighted classes

3.1 The class $\mathcal{E}_{m,F}(H, \Omega)$

Throughout this section we consider $F : \mathbb{R}^- \times \Omega \longrightarrow \mathbb{R}^+$ and μ a nonnegative measure defined in Ω. We will study the following equation

$$\mathcal{H}_{m,F}(.) = \mu \qquad (*)$$

where $\mathcal{H}_{m,F}(u) := F(u(z), z)H_m(u)$. To simplify notations authors in [19] introduced

$$\mathfrak{C}(\mathbb{R}^-) := \{\chi : \mathbb{R}^- \longrightarrow \mathbb{R}^-; \chi \ is \ continuous, \ increasing \ and \ \chi(t) < 0 \ \forall t < 0\}$$

and

$$\mathfrak{D}(\mathbb{R}^-, \Omega) := \{F : \mathbb{R}^- \times \Omega \longrightarrow \mathbb{R}^+; \forall z \in \Omega \, the \ function \ F(., z) \ is \ decreasing \ in \ \mathbb{R}^-\}.$$

Definition 3.1 *For $F \in \mathfrak{D}(\mathbb{R}^-,\Omega)$ and $H \in \mathcal{E}_m(\Omega) \cap \mathcal{MSH}_m(\Omega)$ one can define*

$$\mathcal{E}_{m,F}(H,\Omega) := \{\varphi \in \mathcal{N}_m(H) : \quad \exists \ \mathcal{E}_m^0(H) \ni \varphi_j \searrow \varphi, \\ \sup_{j\geq 1} \int_\Omega \mathcal{H}_{m,F}(\varphi_j) < +\infty\}.$$

To deal with the equation $(*)$ author in [19] proved firstly the following result named as the comparison principle which generlizes basic version of comparison principle in [1, 5, 22]. Namely they proved the following result

Theorem 3.2 *[19] Let $F \in \mathfrak{D}(\mathbb{R}^-,\Omega)$, $u \in \mathcal{N}_m^a(H)$ and $v \in \mathcal{E}_m(H)$. If $\mathcal{H}_{m,F}(u) \leq \mathcal{H}_{m,F}(v)$ then $u \geq v$.*

Based on this result Hbil and Zaway [19] gave a technical demonstration allowing them to solve the equation $(*)$ when the measure μ is assumed to be finite and does not charge any m-polar set. The complete statement of this result appears in the following theorem

Theorem 3.3 *[19]*
Assume that the measure μ is finite with no mass on every m-polar subset Ω and suppose that $\inf_{z\in\Omega} F(t,z) > 0, \forall \ \ t < 0$. Then there exists a function $u \in \mathcal{E}_{m,F}(H,\Omega)$ such that $\mathcal{H}_{m,F}(u) = \mu$. Moreover u is unique.

3.2 Hessian equation on the class $\mathcal{E}_{m,\chi}(\Omega)$

Throughout this section $\chi : \mathbb{R}^- \to \mathbb{R}^-$ will be an increasing function. In [22] Hung introduced the class $\mathcal{E}_{m,\chi}(\Omega)$ to generalize the fundamental weighted energy classes introduced firstly by Benelkourchi, Guedj, and Zeriahi [3]. Such class is defined as follows:

Definition 3.4 *We say that $f \in \mathcal{E}_{m,\chi}(\Omega)$ if there exits $(f_j)_j \subset \mathcal{E}_m^0(\Omega)$ such that $f_j \searrow f$ in Ω and*

$$\sup_{j\in\mathbb{N}} \int_\Omega (-\chi(f_j))H_m(f_j) < +\infty.$$

Remarks 3.5 *It is clear that the class $\mathcal{E}_{m,\chi}(\Omega)$ generalizes all analogous Cegrell classes defined by Lu in [28] and [27]. Indeed*

1. *$\mathcal{E}_{m,\chi}(\Omega) = \mathcal{F}_m(\Omega)$ when $\chi(0) \neq 0$ and χ is bounded.*
2. *$\mathcal{E}_{m,\chi}(\Omega) = \mathcal{E}_m^p(\Omega)$ in the case when $\chi(t) = -(-t)^p$.*
3. *$\mathcal{E}_{m,\chi}(\Omega) = \mathcal{F}_m^p(\Omega)$ in the case when $\chi(t) = -1 - (-t)^p$.*

If we take $m = n$ in all the previous cases we recover the classic Cegrell classes defined in [5] and [6].

Note that in the case $\chi(0) \neq 0$ one has that $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{F}_m(\Omega)$ so the Hessian operator is well defined in $\mathcal{E}_{m,\chi}(\Omega)$ and is with finite total mass on Ω. So in the rest of this study we will always consider the case $\chi(0) = 0$.

In [20] authors prove that the Hessian operator is well defined on $\mathcal{E}_{m,\chi}(\Omega)$. Note that this result was proved also in [22] but with an extra condition $(\chi(2t) \leq a.\chi(t))$ and in [20] we omit that condition and the proof of such result was completely different.

Theorem 3.6 *[20] Assume that $\chi \not\equiv 0$. Then*

$$\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega).$$

So for every $f \in \mathcal{E}_{m,\chi}(\Omega)$, $H_m(f)$ is well defined and $-\chi(f) \in L^1(H_m(f))$.

In this part we will focus on the Hessian equation acting on $\mathcal{E}_{m,\chi}(\Omega)$. The question is to find a sufficient condition to ensure the existence of a solution to the following equation

$$-\chi(u)H_m(.) = \mu \qquad (**)$$

in the class $\mathcal{E}_{m,\chi}(\Omega)$ where μ is a nonnegative measure in Ω. In the particular case when χ is identically equal to -1 the question was solved by [22]. To give a positive answer to the previous question Hbil and zaway [33] used to work in the case when $\chi \in \mathfrak{C}(\mathbb{R}^-)$:

Theorem 3.7 *[33] Let $\chi \in \mathfrak{C}(\mathbb{R}^-)$ and μ a Radon measure. Assume that*

1. *There exists $w \in \mathcal{E}_{m,\chi}(\Omega)$ such that $\mu \leq -\chi(w)H_m(w)$.*
2. *$\mu(\Omega) < +\infty$.*

Then there exists $u \in \mathcal{E}_{m,\chi}(\Omega)$ such that $-\chi(u)H_m(u) = \mu$. Moreover $u \geq w$.

To omit the condition of existence of subsolution in the above theorem Hbil and zaway [33] proceed as follows: Fix $v \in \mathcal{F}_m(\Omega)$, σ a Radon measure with no mass in all m-polar and denote by

$$\mathcal{A}(\sigma, v) = \{\varphi \in \mathcal{E}_m(\Omega) : \sigma \leq -\chi(\varphi)H_m(\varphi)\ ,\ \varphi \leq v\}.$$

Based on the above notations they gave sufficent condition to ensure the existence of solution to $(**)$ as well as an explicit expression of this solution. This is the objective of the following theorem

Theorem 3.8 *[33] Assume that*

1. *$Supp\sigma \Subset \Omega$.*
2. *$SuppH_m(v) \Subset \Omega$ et $H_m(v)$ is carried by $m-$polar subset of Ω.*

Then the function u defined by $u := (sup\{\varphi : \varphi \in \mathcal{A}(\sigma, v)\})^\star$ belongs to $\mathcal{F}_m(\Omega)$ and satisfies $-\chi(u)H_m(u) = \sigma + H_m(v)$.

3.3 Local and Global solution to the Hessian equation

Based on the results obtained in the previous section we show the equivalence between the existence of a global solution and the existence of a local solution to the equation $(**)$. We will give in the following theorem sufficient conditions on the function χ ensuring this equivalence

Theorem 3.9 *[33] Assume that χ is convex and increasing, $\chi(-\infty) > -\infty$ and $\chi(t) < 0$ for every $t < 0$. If $\mu(\Omega) < +\infty$ then the following statement are equivalent:*

- *i) For every $z \in \Omega$ there exists a neighborhood U_z of z and $v_z \in \mathcal{E}_m(U_z)$ such that $\mu \leq H_m(v_z)$ in U_z.*
- *ii) There exists $u \in \mathcal{E}_{m,\chi}(\Omega)$ such that $-\chi(u)H_m(u) = \mu$.*

Definition 3.10 1. *The $m-$capacity of a Borelean subset $E \subset \Omega$ with respect to Ω is denoted by $Cap_m(E, \Omega)$ and defined as*

$$Cap_m(E) =: Cap_m(E, \Omega) = \sup\left\{\int_E H_m(f)\ ,\ f \in \mathcal{SH}_m(\Omega), -1 \leq f \leq 0\right\}.$$

2. *A sequence of functions $(f_j)_j$ defined on Ω is said to be convergent to f with respect to Cap_m, when $j \to +\infty$ if for every compact K of Ω and $\varepsilon > 0$ one has*

$$\lim_{j\to+\infty} Cap_m(\{z \in K : |f_j(z) - f(z)| > \varepsilon\}) = 0.$$

In this part we are interested in the study of the problem linking the convergence in capacity and the convergence of the associated Hessian measures while working on the Cegrell classes $\mathcal{E}_m(\Omega)$ and on the complex energy classes $\mathcal{E}_{m,\chi}(\Omega)$. It is known that capacity convergence plays a crucial role in this study. For more details on the importance of this notion we can refer to [1], [5], [30], [31], [7],[21]. So we continue here these studies in the m-sh case and also on complex energy classes.

3.4 Convergence in $\mathcal{E}_m(\Omega)$

The main idea of this section was inspired by the recent work of V.V. Hung and N.V. Phu in [23] whose results will be essential in the demonstration of various results in this part. We demonstrate in [19] with similar techniques that if $(f_j)_j$ converges in capacity towards a function f then we can control $1_{\{f>-\infty\}}H_m(f)$ by the lower limit of the Hessian measure associated with $(f_j)_j$ (here the sequence as well as its limit are in $\mathcal{E}_m(\Omega)$) more precisely we show the following result

Theorem 3.11 *[20] If f_j is a sequence of functions m-sh belonging to the class $\mathcal{E}_m(\Omega)$ which satisfies $f_j \to f \in \mathcal{E}_m(\Omega)$ in capacity Cap_m. Then*

$$1_{\{f>-\infty\}}H_m(f) \leq \liminf_{j\to+\infty} H_m(f_j).$$

To improve the previous result we will construct classes (by requiring additional conditions) so that we can each time have a result directly linking the limit of the sequence of Hessian measures of $(f_j)_j$ and the Hessian measure of f.

Definition 3.12 *[20] We consider the sets $\mathcal{P}_m(\Omega)$ and $\mathcal{Q}_m(\Omega)$ defined as follows:*

$\mathcal{P}_m(\Omega) = \{f \in \mathcal{E}_m(\Omega)\ ;\exists\ P_1, ..., P_n \textit{ polar in } \mathbb{C}\ /\ 1_{\{f=-\infty\}}H_m(f)(\Omega\backslash P_1\times...\times P_n) = 0\}.$

$\mathcal{Q}_m(\Omega) = \{(f, g) \in (\mathcal{E}_m(\Omega))^2;\ \forall z \in \Omega, \exists V \textit{ a neighborhood of } z \textit{ and } u_V \in \mathcal{E}^a_m(V)\ /\ f+u_V \leq g \textit{ sur } V$

We first demonstrate that the class $\mathcal{P}_m(\Omega)$ is stable by addition and also by maximum with any negative m-sh function. These properties make it possible to improve the result obtained in the theorem 3.11 but for functions in $\mathcal{Q}_m(\Omega)$.

Corollary 3.13 *[20] Let $(f_j)_j \subset \mathcal{E}_m(\Omega)$ be such that $f_j \to f \in \mathcal{E}_m(\Omega)$ in capacity Cap_m. If $f_j, f \in \mathcal{Q}_m(\Omega)$ for all $j \geq 1$. SO*

$$H_m(f) \leq \liminf_{j\to+\infty} H_m(f_j).$$

If we replace the hypothesis $(f_j)_j \subset \mathcal{E}_m(\Omega)$ by the local hypothesis $(f_j)_j \subset \mathcal{F}_m(\Omega)$ then we obtain exactly the convergence of the mass $H_m(f_j)(\Omega)$ to $H_m(f)(\Omega)$. To obtain a stronger version which ensures the continuity of the Hessian operator we had to work in the class $\mathcal{P}_m(\Omega)$. This brings us to the following result

Theorem 3.14 *[20] Let $f_j, g \in \mathcal{E}_m(\Omega)$, $f \in \mathcal{P}_m(\Omega)$ and $D \Subset \Omega$. We suppose that*

- *$f_j \to f$ in capacity Cap_m.*

- *For all $j \geq 1$, $f_j \geq g$ on $\Omega \backslash D$.*

Then $H_m(f_j) \to H_m(f)$ weakly when $j \to \infty$.

Remark 3.15 *The previous theorem represents a different version of Theorem 3.8 in [23]. On the one hand the second hypothesis that we introduce is true outside of a relatively compact set D which is not the case in Theorem 3.8 in [23] because such an hypothesis is assumed to be true on Ω all together. On the other hand the function f is taken in $\mathcal{P}_m(\Omega)$ while in theorem 3.8 in [23] it is enough that $f \in \mathcal{E}_m(\Omega)$.*

3.5 Convergence in $\mathcal{E}_{m,\chi}(\Omega)$

Throughout this section we keep $\chi : \mathbb{R}^- \to \mathbb{R}^-$ an increasing function. In [22] Hung introduced the class $\mathcal{E}_{m,\chi}(\Omega)$ to generalize the fundamental energy classes defined by Benelkourchi, Guedj, and Zeriahi [3]. It should be noted that if we assume that $\chi(0) \neq 0$ then $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{F}_m(\Omega)$ therefore the complex Hessian operator is well defined in $\mathcal{E}_{m,\chi}(\Omega)$ and it has finite total mass on Ω. So here we are only concerned with the case where $\chi(0) = 0$.
Using theorem 3.6 one has that the operator is well defined so here we will study it on $\mathcal{E}_{m,\chi}(\Omega)$. The idea is to use each time the properties of the function χ to obtain characterizations of $\mathcal{E}_{m,\chi}(\Omega)$. For example we show that if the function χ takes the value $-\infty$ in the neighborhood of $-\infty$ then any element of the class $\mathcal{E}_{m,\chi}(\Omega)$ does not charge the m-polars and we actually have an equivalence. This is the objective of the following theorem:

Proposition 3.16 *[20] The following propositions are equivalent:*

1. $\chi(-\infty) = -\infty$
2. $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}^a_m(\Omega)$.

Now we will give a complete characterization of the class $\mathcal{E}^p_m(\Omega)$ introduced by [28] according to the class $\mathcal{N}_m(\Omega)$. To do this we generalize the class of Benelkourchi, Guedj, and Zeriahi [3] $\hat{\mathcal{E}}_\chi(\Omega)$ as follows

Definition 3.17

$$\hat{\mathcal{E}}_{m,\chi}(\Omega) := \left\{ \varphi \in \mathcal{SH}^-_m(\Omega) \, / \, \int_0^{+\infty} t^m \chi'(-t) Cap_m(\{\varphi < -t\}) dt < +\infty \right\}.$$

By demonstrating elementary properties of the class already introduced we were able to give several important characteristics of $\mathcal{E}_{m,\chi}(\Omega)$. The first result obtained in this direction is that any function of this class vanishes on the boundary as soon as we assume that the function χ is strictly negative and also that under this condition we obtain that $\mathcal{E}_{m,\chi}(\Omega)$ is entirely included in the class $\mathcal{N}_m(\Omega)$. More precisely we have

Theorem 3.18 *[20] Suppose that for all $t < 0$ we have $\chi(t) < 0$, then*

$$\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{N}_m(\Omega).$$

Moreover for all $f \in \mathcal{E}_{m,\chi}(\Omega)$ we have

$$\limsup_{z \to w} f(z) = 0, \ \forall w \in \partial\Omega.$$

The techniques used in the demonstration of the result above allow us to give a complete characterization of the class $\mathcal{E}_{m,\chi}(\Omega)$ namely

Corollary 3.19 *[20] If for all $t < 0$; $\chi(t) < 0$ then*

$$\mathcal{E}_{m,\chi}(\Omega) = \left\{ f \in \mathcal{N}_m(\Omega) \,/\, \chi(f) \in L^1(H_m(f)) \right\}.$$

3.6 Extension of the class $\mathcal{E}_{m,\chi}(\Omega)$

As an application of the characterization obtained in the previous section we will extend the theorem 3.14 to the class $\mathcal{E}_{m,\chi}(\Omega)$ which represents a more general result comparing to the work in [23]. More precisely we show these convergence theorems

Theorem 3.20 *[20]*

Suppose that the function χ is continuous, $\chi(-\infty) > -\infty$ and $f, f_j \in \mathcal{E}_m(\Omega)$ for all $j \in \mathbb{N}$. If there exists $g \in \mathcal{E}_m(\Omega)$ such that $f_j \geq g$ on Ω then:

1. *If f_j converges to f in capacity Cap_{m-1} then $\liminf_{j \to +\infty} -\chi(f_j)H_m(f_j) \geq -\chi(f)H_m(f)$.*
2. *If f_j converges to f in capacity Cap_m then $-\chi(f_j)H_m(f_j)$ converges weakly to $-\chi(f)H_m(f)$.*

Finally we deal with the famous problem of extending functions $m - sh$

Definition 3.21 *For $\Omega \Subset \tilde{\Omega} \Subset \mathbb{C}^n$ and $f \in \mathcal{E}_{m,\chi}(\Omega)$, we say $\tilde{f} \in \mathcal{E}_{m,\chi}(\tilde{\Omega})$ is an extension of f if $\tilde{f} \leq f$ on Ω.*

In the case $m = n$ and $\chi \equiv -1$, the problem of existence of extension in this case was studied by Cegrell and Zeriahi [9] in 2003. Then it was generalized by Cegrell, Kolodziej, and Zeriahi [3] for the case of psh functions with weak singularities. The general case when χ is assumed to be arbitrary was solved by Benelkourchi in [2], i.e. on the class $\mathcal{E}_\chi(\Omega)$. Here we show that any function $f \in \mathcal{E}_{m,\chi}(\Omega)$ admits an extension in the send of the definition above.

Theorem 3.22 *[20] Let $\tilde{\Omega}$ be an m-hyperconvex domain such that $\Omega \Subset \tilde{\Omega} \Subset \mathbb{C}^n$. If $\chi(t) < 0$ for all $t < 0$ and $f \in \mathcal{E}_{m,\chi}(\Omega)$ then there exists $\tilde{f} \in \mathcal{E}_{m,\chi}(\tilde{\Omega})$ such as*

$$\int_{\tilde{\Omega}} -\chi(\tilde{f})H_m(\tilde{f}) \leq \int_{\Omega} -\chi(f)H_m(f)$$

and $\tilde{f} \leq f$ on Ω.

3.7 Convergence in capacity and Stability on Cegrell classes

The notion of convergence in capacity plays an important role in the study of different problems relating to the complex Hessian equation and especially in the continuity of the Hessian operator. So in this part we will give equivalences to this notion. For this we will give results on the classes $\mathcal{F}_m^a(\Omega)$ and $\mathcal{N}_m^a(\Omega)$ which are themselves generalizations of the classic case $m = n$. We first extend Cegrells result. We then show this theorem

Theorem 3.23 *[19] Let $(f_j)_j \subset SH_m(\Omega)$ such that*

- *i) $f_0 \leq f_j \leq 0$ for a certain $f_0 \in \mathcal{F}_m^a(\Omega)$;*
- *ii) $f_j \to f$ with respect to the capacity Cap_m on all $E \Subset \Omega$*

then $vH_m(f_j)$ converges weakly to $vH_m(f)$ in Ω and this convergence is uniform with respect to any function m-sh v which is locally uniformly bounded.

This result makes it possible to deduce a relationship between the measure $H_m(u)$ and $H_m(f)$ for any function $f \in \mathcal{N}_m^a(\Omega, u)$. This relationship is the objective of the following theorem

Proposition 3.24 *[19] Let $u \in \mathcal{E}_m(\Omega)$ and $f \in \mathcal{N}_m^a(\Omega, u)$ such that $\int_\Omega(-\varphi)H_m(f) < +\infty$ for some $\varphi \in \mathcal{E}_m^0(\Omega)$. Then*

$$1_{\{f=-\infty\}}H_m(f) = 1_{\{u=-\infty\}}H_m(u) \text{ in } \Omega.$$

Based on the two results already established in this section we give some equivalence to the convergence in capacity namely we proved in [19] the following result

Theorem 3.25 *[19]*

Let Ω be a m-hyperconvex domain bounded in $\mathbb{C}^n$, $u \in \mathcal{E}_m(\Omega)$ and $h \in \mathcal{N}_m^a(\Omega, u)$ which satisfies $\int_\Omega(-\varphi)H_m(h) < +\infty$ for some $\varphi \in \mathcal{E}_m^0(\Omega)$. We assume that $\{f_j\} \subset \mathcal{N}_m^a(\Omega, u)$ such that $f_j \to f_0$ almost everywhere on Ω when $j \to +\infty$ and $f_j \geq h$ in Ω for all $j \geq 0$. Then, the following assertions are equivalent:

(a) $f_j \to f_0$ in capacity Cap_m in Ω;

(b) $\forall r > 0$, we have

$$\lim_{j\to+\infty} \int_\Omega \max\left(\frac{f_j}{r}, \varphi\right) H_m(f_j) = \int_\Omega \max\left(\frac{f_0}{r}, \varphi\right) H_m(f_0).$$

(c) $\forall r > 0$, we have

$$\lim_{j\to+\infty} \int_\Omega \left[\max\left(\frac{g_j}{r}, \varphi\right) - \max\left(\frac{f_j}{r}, \varphi\right)\right] H_m(f_j) = 0,$$

Or $g_j := \left(\sup_{k\geq j} f_k\right)^*$.

The importance of the theorem 3.25 lies in its application to the study of stability of the complex Hessian operator. Indeed, using the equivalences already demonstrated we give a more general version of the stability theorem of Cegrell and Kolodziej [8]. We then show the following results

Lemma 3.26 *Let μ be a nonnegative Borel measure in Ω, $u \in \mathcal{E}_m(\Omega)$, $h \in \mathcal{N}_m^a(\Omega, u)$ satisfying $\int_\Omega (-\varphi) H_m(h) < +\infty$ for some $\varphi \in \mathcal{E}_m^0(\Omega)$. If*

$$H_m(u) \le \mu \le H_m(h),$$

then there exists a unique $f \in \mathcal{N}_m^a(\Omega, u)$ such that $H_m(f) = \mu$ and $f \ge h$ in Ω.

Theorem 3.27 *[19]*

Let $\{\mu_j\}$ be a sequence of positive Borel measures, $u \in \mathcal{E}_m(\Omega)$ and $h \in \mathcal{N}_m^a(\Omega, u)$ such that $\int_\Omega (-\varphi) H_m(h) < +\infty$ for a certain $\varphi \in \mathcal{E}_m^0(\Omega)$. If μ_j converges weakly to a measure μ_0 and

$$H_m(u) \le \mu_j \le H_m(h) \text{ for all } j \ge 0,$$

then there exists a unique $f_j \in \mathcal{N}_m^a(\Omega, u)$ such that $f_j \ge h$, $H_m(f_j) = \mu_j$ and $f_j \to f_0$ in capacity Cap_m in Ω where $H_m(f_0) = \mu_0$.

3.8 Complex Hessian operator associated to m-positif

3.9 m-positif current in the sens of Lu

We recall below the notion of m-positive current introduced by Lu [28].

Definition 3.28 *A current T of bididimensions (p, p), with $1 \le p \le m$ is said to be m-positive if*

$$\alpha_1 \wedge \cdots \wedge \alpha_p \wedge T \ge 0.$$

For any m-positive form $\alpha_1, \cdots, \alpha_p$ of bidegree $(1, 1)$.

Remark 3.29

1. *If $1 \le s < r \le m$, then any $s-$positive current is $r-$positive.*

2. *Recently Dhouib and Elkhadhra [13] gave a new definition of the m-positivity of currents and this notion generalizes the classic positivity introduced by Lelong [25] since 1967 and it will be useful in the following section.*

3. *If T is an m-positive current in the sense of Dhouib and Elkhadhra, then the current $T \wedge \beta^{n-m}$ is m-positive in the sense of Lu.*

In this section we fix T a m-positive current in the sense of Lu such that the following class

$$\mathcal{E}^T_{0,m}(\Omega) := \left\{\varphi \in SH^-_m(\Omega) \cap L^\infty(\Omega);\ \lim_{z\to\partial\Omega\cap Supp\, T}\varphi(z)=0,\ \int_\Omega (dd^c\varphi)^q \wedge T < +\infty\right\}$$

is not empty. The class thus introduced coincides with the standard Cegrell class when $m = n$ and $T = 1$. In a similar way we generalize the Cegrell classes introduced in 1998 and 2004 as follows:

Definition 3.30 *The class $\mathcal{E}^T_{p,m}(\Omega)$ is defined as follows:*

$$\mathcal{E}^T_{p,m}(\Omega) := \left\{\varphi \in SH^-_m(\Omega);\ \exists\ \mathcal{E}^T_{0,m}(\Omega) \ni \varphi_j \searrow \varphi,\ \sup_{j\geq 1}\int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^q \wedge T < +\infty\right\}.$$

If the sequence $(\varphi_j)_j$ associated with φ is chosen such that

$$\sup_{j\geq 1}\int_\Omega (dd^c\varphi_j)^q \wedge T < +\infty,$$

then $\varphi \in \mathcal{F}^T_{p,m}(\Omega)$.

Remark 3.31

1. *In the case $T = 1$, the class $\mathcal{E}^T_{0,m}(\Omega)$ coincides with the class $\mathcal{E}_{0,m}(\Omega)$ introduced by Lu [28].*
2. *If the current T is defined on a neighborhood of Ω, then $\mathcal{E}^T_{0,m}(\Omega)$ contains all bounded functions in $SH^-_m(\Omega)$.*

To be able to rigorously study the classes already introduced we need to demonstrate various properties as well as some useful augmentations. The assessment of these properties, which themselves represent a generalization of the classical case, is the subject of the following proposition:

Proposition 3.32 *[17]*

1. *If $\psi \in SH^-_m(\Omega)$ and $\varphi \in \mathcal{E}^T_{0,m}(\Omega)$ then $\max(\varphi,\psi) \in \mathcal{E}^T_{0,m}(\Omega)$.*
2. *The class $\mathcal{E}^T_{0,m}(\Omega)$ is convex.*
3. *$\mathcal{E}^T_{0,m}(\Omega) \subset \mathcal{F}^T_{p,m}(\Omega) \subset \mathcal{E}^T_{p,m}(\Omega)$.*
4. *$\mathcal{F}^T_{p_1,m}(\Omega) \subset \mathcal{F}^T_{p_2,m}(\Omega)$ for all $p_2 \leq p_1$.*
5. *Suppose $u, v \in \mathcal{E}^T_{0,m}(\Omega)$. If $p \geq 1$ then for all $0 \leq s \leq q$ we have*

$$\begin{aligned}&\int_\Omega (-u)^p (dd^c u)^s \wedge (dd^c v)^{q-s} \wedge T\\ &\leq D_{s,p}\left(\int_\Omega (-u)^p (dd^c u)^q \wedge T\right)^{\frac{p+s}{p+q}} \left(\int_\Omega (-v)^p (dd^c v)^q \wedge T\right)^{\frac{q-s}{p+q}}\end{aligned}$$

where $D_{s,1} = 1$ and $D_{s,p} = p^{\frac{(p+s)(q-s)}{p-1}}$, $p > 1$.

3.10 The class $\mathcal{E}^T_{p,m}(\Omega)$

The objective of this section is to study the operator $(dd^c.)^q \wedge T$ on the class $\mathcal{E}^T_{p,m}(\Omega)$.

To do this we start by showing that this operator is well defined on this class while following the same techniques used by Cegrell [5]. We then obtain the following result

Theorem 3.33 *[17] Let $u \in \mathcal{E}^T_{p,m}(\Omega)$ and $(u_j)_j$ be a sequence of functions m-sh which decreases towards u as in the definition 3.30. Then the sequence $((dd^c u_j)^q \wedge T))_j$ converges weakly to a positive measure μ and this limit is independent of the sequence $(u_j)_j$ chosen. We note $(dd^c u)^q \wedge T := \mu$.*

As the operator is now well defined we moved to study its properties while inspiring properties that the classical Monge-Ampre operator had. We first show a convergence theorem relating to the class $\mathcal{E}^T_{1,m}(\Omega)$. Such a result was established in the case $T = 1$ and $m = n$ by Cegrell [5] and in the case $m = n$ and any T by Zaway [32].

Proposition 3.34 *Let $u \in \mathcal{E}^T_{1,m}(\Omega)$ and $(u_j)_j$ be a decreasing sequence towards u as in the definition of the calsse $\mathcal{E}^T_{1,m}(\Omega)$, then the sequence $(\int_\Omega u_j (dd^c u_j)^q \wedge T)_j$ decreases towards $\int_\Omega u(dd^c u)^q \wedge T$.*

As capacity plays an important role in the study of the operator already cited, we then generalize the notion of Capacity of Bedford Taylor, Lu and Dabbek and Elkhadhra in the following way

$$C_{m,T}(K,\Omega) = C_{m,T}(K) = \sup\left\{\int_K (dd^c v)^q \wedge T,\ v \in SH_m(\Omega, [-1,0])\right\}$$

for any compact K of Ω. Using this capacity we can ask the following question: "Are all functions of the class $\mathcal{E}^T_{p,m}(\Omega)$ quasicontinuous with respect to $C_{m,T}$ as is the case of classical Cegrell class functions $\mathcal{E}_p(\Omega)$?"
In the case $T = (dd^c|z|^2)^{n-m}$, Lu [28] showed that every function m-sh is Cap_m-quasicontinuous. In the general case Dhouib and Elkhadhra showed the quasi-continuity of any function m-sh *bounded* (see Theorem 1 in [13]). However, for the general case we need a sufficient condition to answer positively to the question cited as indicated by the following example:
If Ω is the polydisk of $\mathbb{C}^3$, $T := [z_1 = 0] \wedge dd^c|z|^2$ and $u(z_1, z_2) = \log|z_1|$. The current T is 2-positive, $C_{m,T}(SuppT) > 0$ but the function u is not continuous on the support of T.

To answer to the question already asked we establish the increase which gives a link between the m-capacity of the sets $\{u < 2s\}$ and the mass of the operator.

Proposition 3.35 *Let $u \in \mathcal{E}^T_{p,m}(\Omega)$ and $(u_j)_j \subset \mathcal{E}^T_{0,m}(\Omega)$ decrease towards u on Ω as in the definition 3.30. Then for all $s > 0$ we have*

$$s^{p+q}C_{m,T}(\{u \leq -2s\}, \Omega) \leq \sup_{j\geq 1} \int_\Omega (-u_j)^p (dd^c u_j)^q \wedge T.$$

As a consequence we obtain that for any function $u \in \mathcal{E}^T_{p,m}(\Omega)$ we have $C_{m,T}(\{u = -\infty\} = 0$. Also based on the previous proposition we gave a positive answer to the question already asked, namely the following theorem

Theorem 3.36 *[17] Every $u \in \mathcal{E}^T_{p,m}(\Omega))$ is $C_{m,T}$-quasi-continuous.*

3.11 The class $\mathcal{F}^T_m(\Omega)$

Definition 3.37 *We say that a function $u \in \mathcal{F}^T_m(\Omega)$ if there exists a sequence $(u_j)_j \subset \mathcal{E}^T_{0,m}(\Omega)$ which decreases towards u such that*

$$\sup_j \int_\Omega (dd^c u_j)^q \wedge T < +\infty.$$

This section will then be devoted to studying the class $\mathcal{F}^T_m(\Omega)$. To do this we started to show a comparison result relating to this class namely

Proposition 3.38 *[17] Let $u, v \in \mathcal{F}^T_m(\Omega)$ such that $u \leq v$ on Ω then*

$$\int_\Omega (dd^c v)^q \wedge T \leq \int_\Omega (dd^c u)^q \wedge T.$$

Establishing the above result requires the use of approximations of functions belonging to this class. However, such an approximation existed in the case $m = n$ (see theorem 5.1 in [16]) but with an incomplete proof. For this we first gave a complete demonstration and in a more general case. More precisely we have demonstrated that any function $\varphi \in \mathcal{F}^T_m(\Omega)$ can be approximated by a sequence of functions $(\varphi_j)_j \subset \mathcal{E}^T_{0,m}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Based on this result we deduced a result similar to the theorem 3.36 but on the class $\mathcal{F}^T_m(\Omega)$.

Theorem 3.39 *[17] Any function $\varphi \in \mathcal{F}^T_m(\Omega)$ is $Cap_{m,T}-$quasicontinuous.*

3.12 Xing's inequalities on the class $\mathcal{E}^T_{p,m}(\Omega)$ and $\mathcal{F}^T_m(\Omega)$

Xing's inequalities represented a necessary and indispensable tool in pluripotential theory so it is important to deal with this problem when the functions belong to the class $\mathcal{F}^T_m(\Omega)$ and $mathcalE^T_{p,m}(\Omega)$. To do this we first demonstrate that integration by part remains true on $\mathcal{E}^T_{p,m}(\Omega)$.

Proposition 3.40 *[17] Let $u, w_1, ..., w_{q-1} \in \mathcal{E}^T_{p,m}(\Omega)$ and $S = dd^c w_1 \wedge ... \wedge dd^c w_{q-1} \wedge T$. SO*

$$\int_\Omega v dd^c u \wedge S = \int_\Omega u dd^c v \wedge S.$$

Moreover if we assume that $u \le v$ on Ω, then for all $p > 0$ and $h \in \mathcal{E}^T_{0,m}(\Omega) \cap \overline{\mathcal{C}}(\Omega)$

$$\int_\Omega (-h)(dd^c v)^q \wedge T \le \int_\Omega (-h)(dd^c u)^q \wedge T.$$

Using the previous proposition we show the first equality of type Xing. More precisely we have

Theorem 3.41 *[17]*

Let $0 < p \le 1$ and $u, v \in \mathcal{E}^T_p(\Omega)$ such that $(dd^c v)^q \wedge T$ does not load the $(m, T)-$pluripole then

$$\int_{\{v<u\}} (dd^c u)^q \wedge T \le \int_{\{v<u\}} (dd^c v)^q \wedge T$$

As a consequence we obtain

Theorem 3.42 *[17] Let $u_1, u_2, \cdots, u_p \in \mathcal{F}^T_m(\Omega)$ and $h \in \mathcal{E}^{m,T}_0(\Omega)$ then we have:*

$$\int_\Omega -h dd^c u_1 \wedge \cdots \wedge dd^c u_p \wedge T \le \left(\int_\Omega -h(dd^c u_1)^p \wedge T \right)^{\frac{1}{p}} \cdots \left(\int_\Omega -h(dd^c u_p)^p \wedge T \right)^{\frac{1}{p}}.$$

3.13 m-positive currents in the sense of Dhouib and Elkhadhra

In this part we study the complex Hessian operator associated with a m-positive current in the sense of Dhouib and Elkhadhra [13]. This definition was given to generalize the standard positivity of forms and currents.

Definition 3.43 *[13] Let φ be a $(p, p)-$real form defined on Ω and m be an integer such that $p \le m \le n$. The form φ is said to be strongly m-positive on Ω if it can be written in the following form*

$$\varphi = \sum_{k=1}^{N} \lambda_k \alpha^k_1 \wedge \cdots \wedge \alpha^k_p$$

where $\alpha^k_1, \cdots, \alpha^k_p$ are m-positive forms on Ω and $\lambda_k \ge 0$.
By duality a current T of two dimensions $(n-p, n-p)$ is said to be m-positive if we have

$$\langle T, \beta^{n-m} \wedge \varphi \rangle \ge 0$$

For any strongly m-positive φ form of bidegrees $(m-p, m-p)$.

Throughout this part T will be an m-positive closed current of bidegree (p,p) in the sense of the definition 3.43. In [13] the authors developed a pluripotential theory relating to T. Based on the definition 3.43, they defined the complex Hessian operator associated with a closed m-positive current to generalize the work of Bedford and Taylor[1], Blocki [4], and Lu [27]. They showed that the operator $(dd^c.)^p \wedge T \wedge \beta^{n-m}$ is well defined on the set of functions m-sh which are possibly bounded in the neighborhood of $\partial\Omega \cap SuppT$. The essential tool in their study is the convergence in capacity $cap_{m,T}$ defined according to the Hessian measure associated with T as follows

Definition 3.44 *For any compact K of Ω the m-capacity of K associated with T denoted by $cap_{m,T}(K)$ is defined as being*

$$cap_{m,T}(K,\Omega) = cap_{m,T}(K) := \sup\{\int_K (dd^c v)^{m-p} \wedge T \wedge \beta^{n-m}, v \in SH_m(\Omega), 0 \le v \le 1\},$$

and for all $E \subset \Omega$, $cap_{m,T}(E,\Omega) = \sup\{cap_{m,T}(K), K \text{ a compact of } \Omega\}$.

We are interested here in studying problems relating to convergence in capacity $cap_{m,T}$.
We first show that the convergence of a sequence of locally bounded m-sh functions $(u_j)_j$ towards a function u, implies its convergence in capacity $cap_{m,T}$. This generalizes the known result in the standard case which was given by Bedford and Taylor [1] for the case $m = n$ and $T = 1$ and by Lu [28] when $T = 1$ and any m. This result is given as follows

Theorem 3.45 *[18]*
If $u_j, u \in SH_m(\Omega) \cap L^\infty_{loc}(\Omega)$ such that $u_j = u$ on a neighborhood of $\partial\Omega$ and u_j decreases towards u, then for all $\delta > 0$ we have:

$$\lim_{j\to+\infty} cap_{m,T}\{z \in \Omega, u_j(z) > u(z) + \delta\} = 0.$$

Based on the quasicontinuity theorem demonstrated by Dhouib el Elkhadhra in [13],they gave a link between the convergence in capacity and the convergence of the Hessian measure and always when these two notions are relative to the current T.

Theorem 3.46 *[13]*
Let $(u_j)_j$ be a sequence of functions m-sh locally uniformly bounded on Ω and $u \in SH_m(\Omega) \cap L^\infty_{loc}($ Omega$)$.

1. *If u_j converges to u in capacity $cap_{m,T}$ then for all $E \Subset \Omega$, the sequence $(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$ weakly converges to $(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m}$.*

2. *It is assumed that it exists $E \Subset \Omega$ such that $\forall j,\ u_j = u$ on $\Omega \setminus E$ and that the sequences $u(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$, $u_j(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m}$ and $u_j(dd^c u_j)^{m-p} \wedge T \wedge \beta^{n-m}$ converge weakly to $u(dd^c u)^{m-p} \wedge T \wedge \beta^{n-m}$ then u_j converges to u with respect to the capacity $cap_{m,T}$ on E.*

If we replace the current T by a sequence of currents which converge towards it then we obtain a generalization of the Elkhadhra theorem [15] which was demonstrated in the case $m = n$.

Theorem 3.47 *[18]*

Let T and $(T_k)_k$ be closed m-positive currents of two dimensions (p, p) on Ω such that T_k converges in the direction of the currents towards T in Omega. Let u and u_k be $m-sh$ functions locally uniformly bounded on Ω such that $u_k \to u$ in capacity $cap_{m,T}$ on all $E \Subset \Omega$. If we assume that

$$||T_k \wedge \beta^{n-m}|| \ll cap_{m,T}$$

on all $E \Subset \Omega$ uniformly when $k \to \infty$,

then $dd^c u_k \wedge T_k \wedge \beta^{n-m} \to dd^c u \wedge T \wedge \beta^{n-m}$ weakly Ω.

4 Open Problem

Following the results cited above several open problems can be noticed:

- (P1): Following the result found in the Lemma 3.26, the right hand side of the assumption was given by a function $h \in \mathcal{N}^a_m(\Omega, u)$. So one can ask the following open question " If we assume that the right hand side is a measure carried by an analytical set, can we prove that the existence of a sub-solution gives a solution?". The motivation to ask such question follows from a remark given by Dinew and Chinh in [11] where they gave an example of $m-$subharmonic function solution of the Hessian equation with right hand side carried by analytic set which is hard to find in the case of plurisubharmonic function.

- (P2): Can we characterize the (m, T)-polar sets as in the classical case $m = n$ and $T = 1$ where negligible sets are pluripolar [1].

- (P3): It is known that m-polar sets are given by an m-subharmonic functions. Can we characterize them using function in the weighted energy class as it is true when $m = n$ (See [3]).

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