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Exploring the weighted harmonic mean transformation to generate new probability distributions

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Abstract

This paper introduces a novel approach that utilizes the weighted harmonic mean in combination with multiple univariate cumulative distribution functions to create a fresh univariate cumulative distribution. In addition to an innovative expression, it exhibits direct ordering properties involving traditional mixtures of univariate cumulative distribution functions. After establishing the theoretical foundation, we leverage these findings to develop new families of trigonometric distributions. We highlight a special case that extends the exponentiated distribution's scope and demonstrates its superior performance in data fitting compared to fair competitors like the exponential, gamma, exponentiated exponential, and sine exponential distributions. In summary, our methodology offers a versatile framework for modeling and understanding a wide array of statistical phenomena.

Keywords: *Mixture Distributions, Weighted Harmonic Mean, Trigonometric Family of Distributions.*

2010 Mathematics Subject Classification: 60E05, 26D15.

1 Introduction

The study of cumulative distribution functions (CDFs) plays a pivotal role in statistical analysis, probability theory, and data modeling. These functions provide essential insights into the probabilities associated with random variables and underpin various statistical methodologies. In particular, the mixture CDFs that combine multiple univariate CDFs have long been a cornerstone in statistics. The most famous examples are the weighted arithmetic and geometric means of univariate CDFs, as described below.

- Let n be a positive integer, $F_1(x), \ldots, F_n(x)$ be n univariate CDFs of absolutely continuous distributions, and $\alpha_1, \ldots, \alpha_n$ be n non-negative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. Then the following weighted arithmetic mean of $F_1(x), \ldots, F_n(x)$ is a valid CDF:

$$U_n(x) = \sum_{i=1}^{n} \alpha_i F_i(x), \quad x \in \mathbb{R}. \tag{1}$$

- In a similar framework, with the use of products, the following weighted geometric mean of $F_1(x), \ldots, F_n(x)$ is a valid CDF:

$$V_n(x) = \prod_{i=1}^{n} [F_i(x)]^{\alpha_i}, \quad x \in \mathbb{R}. \tag{2}$$

Both approaches offer a flexible framework for capturing a wide range of distribution shapes. The literature on this topic is vast. A short list of references is [5], [10], [12], [20], [7], [11], [3], [2], [15], [13], [14] and [1]. However, as data sets become increasingly diverse and complex, there is a growing need for novel approaches that go beyond conventional mixture CDFs.

In this paper, we introduce a new approach that leverages the concept of the weighted harmonic mean and combines it with several univariate CDFs to construct a new, innovative univariate CDF. It offers a real alternative to traditional mixture CDFs, presenting a fresh perspective on modeling complex data distributions. One of the key advantages of our approach is its direct ordering properties compared to traditional mixture CDFs. After establishing the theoretical basis, we use our findings to create new kinds of trigonometric families of univariate absolutely continuous distributions, i.e., those generated by CDFs involving trigonometric functions $(\sin, \cos, \tan, \ldots)$. Creating such trigonometric families holds promise for capturing complex patterns in data analysis. See [18], [19], [16] and [17]. We highlight a special case extending the scope of the exponentiated distribution. It is illustrated by a statistical application, revealing that it can be more accurate in the data fitting objective than strong direct competitors such as the exponential, gamma, exponentiated exponential (exp.exponential), and sine exponential distributions. Thus, our methodology promises to provide a versatile framework for modeling and understanding a wide range of statistical phenomena.

The plan is as follows: Section 2 is devoted to the main results. Applications are given in Section 3. An open problem is formulated in Section 4.

2 Results

The main findings are presented in this section.

2.1 Weighted Harmonic Mean Approach

In order to motivate our approach, a retrospective on the weighted harmonic mean transformation is necessary. As a first presentation, the weighted harmonic mean is a statistical measure that combines the concepts of harmonic mean and weighted average. It is used to calculate a weighted average in situations where the data has varying degrees of importance. For a positive integer n, n data $x_1, \ldots, x_n$ and n non-negative real numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum_{i=1}^{n} \alpha_i = 1$, the weighted harmonic mean of $x_1, \ldots, x_n$ is indicated as

$$w_n = \left[\sum_{i=1}^{n} \frac{\alpha_i}{x_i} \right]^{-1}.$$

Thus, it is useful to give more importance or weight to certain data while calculating the mean, especially when dealing with values that are reciprocals or rates, such as speed, efficiency, or ratios. By defining the weighted arithmetic and geometric means of $x_1, \ldots, x_n$ as

$$u_n = \sum_{i=1}^{n} \alpha_i x_i$$

and

$$v_n = \prod_{i=1}^{n} x_i^{\alpha_i},$$

respectively, the following inequalities hold:

$$w_n \leq v_n \leq u_n. \tag{3}$$

See [4] for the details. As detailed in the introduction, the weighted arithmetic and geometric means of multiple univariate CDFs have been the object of several studies in the literature. However, to the best of our knowledge, there is no evocation of the use of the weighted harmonic means of CDFs. In this paper, we aim to fill this gap by introducing and examining the concept of the weighted harmonic mean applied to multiple univariate CDFs, thereby contributing to the broader understanding of statistical aggregation methods in probability theory.

2.2 Main Result

The weighted harmonic mean of multiple univariate absolutely continuous CDFs is described below.

Proposition 2.1 *Let n be a positive integer, $F_1(x), \ldots, F_n(x)$ be n univariate CDFs of absolutely continuous distributions and $\alpha_1, \ldots, \alpha_n$ be n non-negative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$. Then the weighted harmonic mean of $F_1(x), \ldots, F_n(x)$ given as*

$$W_n(x) = \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-1}, \quad x \in \mathbb{R}, \tag{4}$$

is a valid CDF.

Proof. Since, for any $i = 1, \ldots, n$, $\lim_{x \to -\infty} F_i(x) = 0$ with $a_i \geq 0$, we have

$$\lim_{x \to -\infty} W_n(x) = \lim_{x \to -\infty} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-1} = \lim_{y \to +\infty} \frac{1}{y} = 0.$$

On the other hand, since, for any $i = 1, \ldots, n$, $\lim_{x \to +\infty} F_i(x) = 1$ with $\sum_{i=1}^{n} \alpha_i = 1$, we have

$$\lim_{x \to +\infty} W_n(x) = \lim_{x \to +\infty} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-1} = \left[\sum_{i=1}^{n} \alpha_i \right]^{-1} = 1.$$

Furthermore, by using standard differentiation rules, we establish that

$$W_n(x)' = \left\{ \sum_{i=1}^{n} \frac{\alpha_i f_i(x)}{[F_i(x)]^2} \right\} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-2},$$

where $f_i(x)$ is the probability density function (PDF) associated with $F_i(x)$. Since, for any $i = 1, \ldots, n$, $\alpha_i \geq 0$, $f_i(x) \geq 0$ and $F_i(x) \geq 0$, we have $W_n(x)' \geq 0$. As a result, $W_n(x)$ is non-decreasing. We conclude that $W_n(x)$ is a valid CDF. The proof ends. □

Remark 2.2 *For the non-decreasing property of $W_n(x)$, the following development is also valid: For any $x \leq y$ and $i = 1, \ldots, n$, we have $F_i(x) \leq F_i(y)$, and since $\alpha_i \geq 0$, we have $\alpha_i / F_i(y) \leq \alpha_i / F_i(x)$, so $\sum_{i=1}^{n} [\alpha_i / F_i(y)] \leq \sum_{i=1}^{n} [\alpha_i / F_i(x)]$, which implies that $W_n(x) \leq W_n(y)$. The non-decreasing property of $W_n(x)$ is proved. The interest of this alternative proof is that it is also valid for univariate CDFs of discrete distributions, beyond the absolutely continuous case.*

To the best of our knowledge, the CDF in Equation (4) has never been studied. It has the advantage of being very general; we can tune $F_1(x), \ldots, F_n(x)$ and $\alpha_1, \ldots, \alpha_n$ to make it appropriate for a given modeling problem. Also, the corresponding PDF is quite simple; it is

$$g_n(x) = \left\{ \sum_{i=1}^{n} \frac{\alpha_i f_i(x)}{[F_i(x)]^2} \right\} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-2}, \quad x \in \mathbb{R}, \tag{5}$$

which can also be written as the following finite linear combination:

$$g_n(x) = \sum_{i=1}^{n} \alpha_i h_i(x), \quad x \in \mathbb{R},$$

where, for any $i = 1, \ldots, n$,

$$h_i(x) = \frac{f_i(x)}{[F_i(x)]^2} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-2}.$$

Similarly, other functions of interest can be expressed. For instance, the survival, hazard and reversed hazard rate functions are given as

$$K_n(x) = 1 - \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-1}, \quad x \in \mathbb{R},$$

$$q_n(x) = \left\{ \sum_{i=1}^{n} \frac{\alpha_i f_i(x)}{[F_i(x)]^2} \right\} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-1} \left\{ \sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} - 1 \right\}^{-1}, \quad x \in \mathbb{R}$$

and

$$r_n(x) = \left\{ \sum_{i=1}^{n} \frac{\alpha_i f_i(x)}{[F_i(x)]^2} \right\} \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x)} \right]^{-1}, \quad x \in \mathbb{R},$$

respectively. One limitation of our approach, however, is the determination of the quantile function based on the CDF in Equation (4), which can be difficult to exhibit analytically. Indeed, it is the function $S_n(x)$ satisfying the following nonlinear equation: $W_n[S_n(x)] = x$, i.e., such that

$$\sum_{i=1}^{n} \frac{\alpha_i}{F_i[S_n(x)]} = \frac{1}{x},$$

which must take into account the simultaneous actions of $F_1(x), \ldots, F_n(x)$.

Another interest concerns a manageable ordering of CDFs, as formulated in the next result.

Proposition 2.3 *Let us consider $U_n(x)$, $V_n(x)$ and $W_n(x)$ as defined in Equations* (1), (2), *and* (4), *respectively. Then, for any $x \in \mathbb{R}$, the following ordering of CDFs holds:*

$$W_n(x) \leq V_n(x) \leq U_n(x).$$

The proof is an immediate consequence of the inequalities in Equation (3).

In the sense of Proposition 2.3, for the same univariate CDFs $F_1(x), \ldots, F_n(x)$, the corresponding weighted harmonic mean offers a real modeling alternative to the corresponding weighted arithmetic and geometric means. The potential of our method in applied sciences is considerable given the diverse applications of the weighted arithmetic and geometric means of multiple univariate CDFs.

3 Applications

This section is devoted to some applications of the proposal.

3.1 New Trigonometric Families of Distributions

In this part, we show how the suggested weighted harmonic transformation can be used to generate new and practical distributions. To this end, we focus on the case $n = 2$ and we consider $\alpha \in [0, 1]$, such that $\alpha_1 = \alpha$ and $\alpha_2 = 1 - \alpha$. The corresponding weighted harmonic mean of $F_1(x)$ and $F_2(x)$ becomes

$$W_2(x) = \left(\frac{\alpha}{F_1(x)} + \frac{1-\alpha}{F_2(x)} \right)^{-1}, \quad x \in \mathbb{R}.$$

After some arrangements, we can also write it as

$$W_2(x) = \frac{F_1(x)F_2(x)}{F_1(x) - \alpha[F_1(x) - F_2(x)]}, \quad x \in \mathbb{R}.$$

Several strategies are possible for the choices of $F_1(x)$ and $F_2(x)$, including the CDFs of various lifetime distributions (see [21]). Obviously, $W_2(x)$ corresponds to $F_1(x)$ for $\alpha = 1$ and to $F_2(x)$ for $\alpha = 0$.

Based on $W_2(x)$, an immediate example of simple family of distributions that can be derived is obtained by fixing a single CDF, say $F(x)$, and choosing $F_1(x) = F(x)$ and $F_2(x) = [F(x)]^{\theta+1}$ with $\theta > 0$, which gives

$$W_2(x) = \frac{[F(x)]^{\theta+1}}{1 - \alpha\{1 - [F(x)]^{\theta}\}}, \quad x \in \mathbb{R}.$$

The corresponding family appears to be a modification of the famous Marshall-Olkin family, as described in [9]. Further work in this direction is possible.

In what follows, we propose to use our approach to contribute to the development of the trigonometric families. We fix a single CDF, say $F(x)$, put $F_1(x) = F(x)$, define $F_2(x)$ as various CDFs of existing trigonometric families, and list some new families derived from the definition of $W_2(x)$.

- By choosing $F_2(x) = \sin[(\pi/2)F(x)]$, which is the CDF of the sine generated family (see [19]), we create the weighted harmonic sine generated family by the following CDF:

$$W_2(x) = \left(\frac{\alpha}{F(x)} + \frac{1-\alpha}{\sin[(\pi/2)F(x)]}\right)^{-1}, \quad x \in \mathbb{R}.$$

  We can also write it as

$$W_2(x) = \frac{F(x)\sin[(\pi/2)F(x)]}{F(x) - \alpha\{F(x) - \sin[(\pi/2)F(x)]\}}, \quad x \in \mathbb{R} \tag{6}$$

  or as the following sine and sine cardinal expression:

$$W_2(x) = \frac{\sin[(\pi/2)F(x)]}{1 - \alpha\{1 - (\pi/2)\,\mathrm{sinc}[(\pi/2)F(x)]\}}, \quad x \in \mathbb{R},$$

  where $\mathrm{sinc}(x) = \sin(x)/x$ for $x \neq 0$ and $\mathrm{sinc}(x) = 1$ for $x = 0$.

- By selecting $F_2(x) = \tan[(\pi/4)F(x)]$, which is the CDF of the tangent generated family (see [18]), we introduce the weighted harmonic tangent generated family by the following CDF:

$$W_2(x) = \left(\frac{\alpha}{F(x)} + \frac{1-\alpha}{\tan[(\pi/4)F(x)]}\right)^{-1}, \quad x \in \mathbb{R}.$$

  We can also write it as

$$W_2(x) = \frac{F(x)\tan[(\pi/4)F(x)]}{F(x) - \alpha\{F(x) - \tan[(\pi/4)F(x)]\}}, \quad x \in \mathbb{R}.$$

- By choosing $F_2(x) = 1 - \cos[(\pi/2)F(x)]$, which is the CDF of the cosine generated family (see [16]), we define the weighted harmonic cosine generated family by the following CDF:

$$W_2(x) = \left(\frac{\alpha}{F(x)} + \frac{1-\alpha}{1-\cos[(\pi/2)F(x)]} \right)^{-1}, \quad x \in \mathbb{R}.$$

We can also write it as

$$W_2(x) = \frac{F(x)\{1-\cos[(\pi/2)F(x)]\}}{F(x) - \alpha\{F(x) - 1 + \cos[(\pi/2)F(x)]\}}, \quad x \in \mathbb{R}.$$

- By selecting $F_2(x) = \sec[(\pi/3)F(x)] - 1$, which is the CDF of the secant generated family (see [17]), we create the weighted harmonic secant generated family by the following CDF:

$$W_2(x) = \left(\frac{\alpha}{F(x)} + \frac{1-\alpha}{\sec[(\pi/3)F(x)] - 1} \right)^{-1}, \quad x \in \mathbb{R},$$

where $\sec(x) = 1/\cos(x)$. We can also write it as

$$W_2(x) = \frac{F(x)\{\sec[(\pi/3)F(x)] - 1\}}{F(x) - \alpha\{F(x) - \sec[(\pi/3)F(x)] + 1\}}, \quad x \in \mathbb{R}.$$

To the best of our knowledge, none of these trigonometric families have been the subject of research, despite evident interest. This statement is illustrated in the next part.

3.2 Illustration

Let us consider the weighted harmonic sine generated family as indicated by the CDF in Equation (6) and apply it with the exponential distribution for the baseline. That is, we consider the following CDF: $F(x) = 1 - e^{-\lambda x}$ for $x > 0$ and with $\lambda > 0$, and $F(x) = 0$ otherwise, and, after the use of some trigonometric formulas, the CDF in Equation (6) becomes

$$W_2(x) = \frac{(1-e^{-\lambda x})\cos[(\pi/2)e^{-\lambda x}]}{1 - e^{-\lambda x} - \alpha\{1 - e^{-\lambda x} - \cos[(\pi/2)e^{-\lambda x}]\}}, \quad x > 0,$$

where $\alpha \in [0, 1]$, and $W_2(x) = 0$ for $x \leq 0$.

Let us call the related distribution the weighted harmonic sine exponential (WHSE) distribution. It is a two-parameter lifetime distribution that realizes a weighted harmonic mean tradeoff between the exponential distribution and the sine exponential distribution, as indicated in [8]; the exponential distribution is obtained by taking $\alpha = 1$ and the sine exponential distribution comes by choosing $\alpha = 0$.

By differentiation (or using Equation (5)), the corresponding PDF is obtained as

$$g_2(x) = \frac{\alpha\lambda e^{-\lambda x}\{\cos[(\pi/2)e^{-\lambda x}]\}^2 + (1-\alpha)(1-e^{-\lambda x})^2(\pi/2)\lambda e^{-\lambda x}\sin[(\pi/2)e^{-\lambda x}]}{[1-e^{-\lambda x}-\alpha\{1-e^{-\lambda x}-\cos[(\pi/2)e^{-\lambda x}]\}]^2}, \quad x > 0,$$

and $g_2(x) = 0$ for $x \leq 0$. This PDF function can serve as the mathematical model that describes the likelihood of observing the given data. In maximum likelihood estimation, the goal is to find the parameters of the PDF that maximize the likelihood of the observed data. The aim is to determine the most probable set of parameters that generated the observed data, making it a fundamental tool in statistical modeling and data analysis.

With this approach in mind, we illustrate the interest of the WHSE distribution by fitting a real data set and compare the obtained results with those of its serious competitors: the exponential, gamma, exp.exponential, and sine exponential distributions, which are all valuable generalizations of the exponential distributions (without power shape parameter in the main exponential term for fairness). More specifically, the breaking stress of carbon fibers data set from sample 100 observations is adapted for data analysis. This data set was used by [6] to illustrate the usefulness of the sine exponential distribution. The data are as follows: 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.
Statistical model selection tools such as the maximized log-likelihood ($LogL$), Akaike information criterion (AIC), Bayesian information criterion (BIC), and Kolmogorov-Smirnov ($K-S$) and Cramér-von Mises (W^*) test statistics with their corresponding p-values are considered for model comparison. The suitable model for fitting the data set under consideration is associated with the one having the maximized log-likelihood value and the smallest value in terms of the AIC, BIC, $K-S$, and W^* with the highest p-values. The summary results and the graphical PDF fit of the distributions for the data set are presented in Table 1 and Figure 1, respectively.

Table 1: Summary of the fitting results

Distributions	Parameter estimates	$LogL$	AIC	BIC	$K-S$ (p-value)	W^* (p-value)
WHSE	$\alpha = 0.0355$	-142.8965	289.793	295.0034	0.0569	0.0605
	$\lambda = 1.7316$				(0.9024)	(0.8121)
Sine exponential	$\lambda = 0.2188$	-191.201	384.4021	387.0074	0.3110	3.2185
					(7.9×10^{-9})	(1.5×10^{-8})
Exp.exponential	$\lambda = 0.9869$	-146.1823	296.3646	301.5749	0.1077	0.2292
	$\theta = 7.7897$				(0.196)	(0.2174)
Gamma	$\alpha = 5.9545$	-143.2336	290.4673	295.6776	0.0934	0.1501
	$\lambda = 2.2715$				(0.3473)	(0.3893)
Exponential	$\lambda = 0.3814$	-196.3709	394.7417	397.3469	0.3206	3.4340
					(2.36×10^{-9})	(4.56×10^{-9})

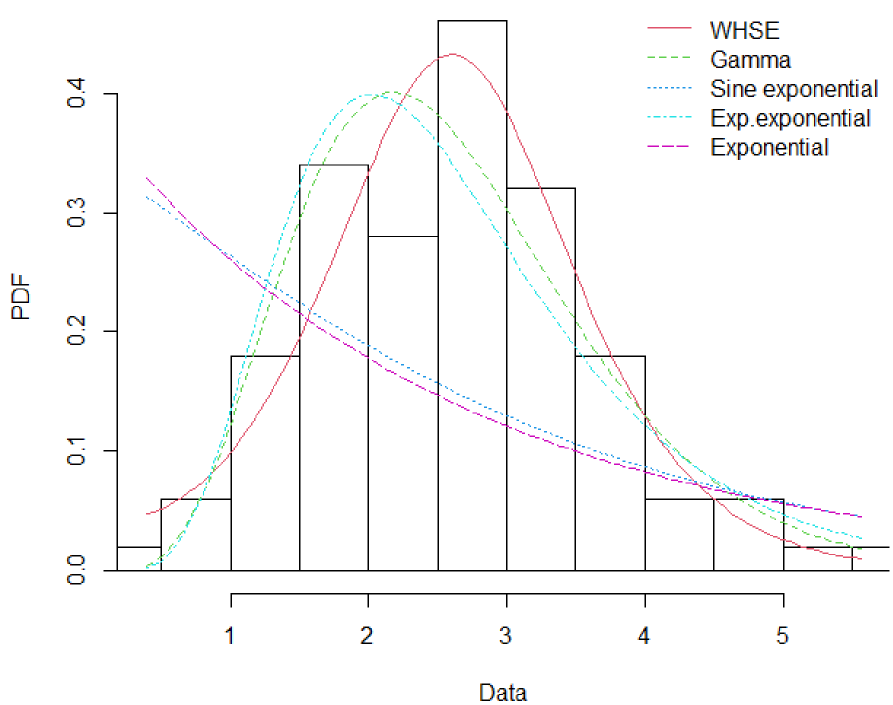

Figure 1: Estimated PDF fits of the distributions for the considered data set

Based on the criteria for model selection, the fits in Table 1 are obviously in support of the WHSE distribution, having the maximized log-likelihood value and the smallest value in terms of the AIC, BIC, $K-S$, and W^* with the highest p-values. On the other hand, the graphical illustration of the estimated PDF in Figure 1 shows that the fit of the WHSE distribution matches closer to the empirical PDF of the data set than the competing distributions, thus validating the superiority of the WHSE distribution over the competing distributions. On the other hand, the results of the exponential and sine exponential are not good, while those of the proposed tradeoff harmonic mean are suitable, confirming the interest of our approach.

4 Open Problem

Based on Proposition 2.1, a question naturally arises: Under what assumptions can we extend this result to the bivariate (or multivariate) case? That is, let n be a positive integer, $F_1(x, y), \ldots, F_n(x, y)$ be n bivariate CDFs of absolutely continuous distributions, and $\alpha_1, \ldots, \alpha_n$ be n non-negative real numbers such that $\sum_{i=1}^{n} \alpha_i = 1$, and

$$Z_n(x, y) = \left[\sum_{i=1}^{n} \frac{\alpha_i}{F_i(x, y)} \right]^{-1}, \quad (x, y) \in \mathbb{R}^2.$$

Under what assumptions $Z_n(x, y)$ is a valid CDF? The answer seems not to be immediate and deserves a complete study that we postpone for future research.

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