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On Orthogonal Polynomials and Norm-Attainable Operators in Hilbert Spaces

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Abstract

In this paper, we explore the intricate connection between orthogonal polynomials and norm attainable operators within a Hilbert space. We give examples of this relationship by associating Hermite, Laguerre, and Jacobi orthogonal polynomials with second-order differential operators. These associations are substantiated by demonstrating various properties, notably the assertion that if a second-order differential operator is expressed in a specific form with continuous coefficients and smooth functions, then it qualifies as self-adjoint. Furthermore, we establish that when the coefficients of such an operator are real-valued functions ensuring positivity, an eigenvalue problem associated with it yields eigenfunctions forming an orthonormal basis, and consequently, the operator is normal.

Keywords: *Differential Operator, Hilbert Space, Orthogonal Polynomial, Spectral Analysis, Norm-attainability.*
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1 Introduction

This work explores the profound domain of differential operators [6] within the context of Hilbert spaces [9], shedding light on their intricate mathematical properties and applications [1]. Specifically, it delves into the intriguing relationship between differential operators [16] and orthogonal polynomials [2], unveiling the role of spectral analysis in understanding their behaviorn [13]. This investigation aims to elucidate the concept of norm-attainability (see [11]-[15] and the references therein) and its significance in the context of these operators, with a focus on spectral decomposition techniques [3]. By examining these fundamental aspects, the study contributes to the broader field of functional analysis, offering valuable insights into the spectral properties of differential operators in Hilbert spaces. In our research, we delve into a mathematical framework centered around a non-decreasing function $\alpha(x)$ defined on the interval $[a, b]$, and the behavior of measurable functions $g(x)$ in the Lebesgue space $L^p(X, \xi_1, \mu)$ for x in the domain X [18]. We assume the existence of $\int_a^b |g(x)|^p d\alpha(x)$, with a special focus on the case where $\alpha(x)$ equals x and p equals 2, denoting this space as $L^2(a, b)$. We introduce an inner product [4], $\langle g_1, g_2 \rangle$, exploiting the Hilbert space structure [20], which is applicable for monotonic $\alpha(x)$. This concept extends to cases with bounded variation [6]. When the interval limits, a and b, are finite, and $\alpha(x)$ is bounded, and $g(x) \in L^p(x)$ is continuous, we employ an integration formula. It allows us to interpret $d\alpha(x)$ as a representation of continuous or discontinuous mass distribution within $[a, b]$. If $\alpha(x)$ is absolutely continuous, we rewrite certain expressions using a weight function $w(x)$. We also quantify total mass within subintervals [8]. In our finite interval scenario, we consider $d\alpha(x)$ or $w(x)$ as fixed distributions in the vector space $L^2_\alpha(a, b)$. Subsequently, our focus shifts to the examination of linear differential operators that operate within the domains of continuous functions denoted as $C^0([0, 1])$ and infinitely differentiable functions [9] represented as $C^\infty([0, 1])$. The methodology employed in the presented mathematical results encompasses a comprehensive approach rooted in functional analysis and linear differential equations [6]. It commences with the definition of the operator T, a second-order linear differential operator, and establishes the framework for function spaces, emphasizing infinite differentiability [10]. Essential components of the methodology encompass the introduction of inner products and norms in function spaces, adjoint operators, and the exploration of eigenvalue problems for T, substantiating the existence of eigenvalues and eigenfunctions [16]. The concept of diagonalizability, orthogonal eigenspaces, and operator properties, such as normality and self-adjointness, is scrutinized, with rigorous mathematical proofs forming a pivotal part of the analysis [17]. Additionally, the discussion incorporates limit analysis and convergence considerations, underscoring the importance of chosen function spaces throughout the analysis [20]. Ultimately, the methodology culminates in the summation of results and their implications within the context of the operator T and function spaces [7].

2 Preliminaries

For a better understanding of this work, we outline the basic definitions that are key to this note on orthogonal polynomials and norm-attainability.

Definition 2.1 *([8]) We denote differential operators as D, often defined on a suitable domain $\Omega \subseteq \mathcal{R}$. These operators may involve various orders of differentiation and are considered densely defined on the Hilbert space.*

Definition 2.2 *([4]) The spectral analysis of T^*, where T^* is the adjoint of T, plays a central role in this research. The spectrum of T^*, denoted $\sigma(T^*)$, characterizes the eigenvalues and essential spectrum of the operator.*

Definition 2.3 *([2]) The study extensively employs orthogonal polynomials, often associated with the Sturm-Liouville theory. These polynomials form a complete orthogonal basis in $L^2(\Omega)$, where $L^2(\Omega)$ represents the space of square-integrable functions on Ω.*

Definition 2.4 *([13]) Normality of operators is a critical concept in this context. A densely defined operator T is said to be normal if $TT^* = T^*T$, where T^* denotes the adjoint of T. Normal operators have several intriguing spectral properties.*

At this point, we proceed to give the main results of this paper. These results are restricted to the space of all norm-attainable real-valued functions. An operator ϕ is said to be norm-attainable [5] if there exists a unit vector ξ in the domain of ϕ such that $\|\phi(\xi)\| = \|\phi\|$. The space of all norm-attainable operators is a normed linear space. For details on norm-attainability, see [11]-[15] and the references therein.

3 Main results

We provide the main results of this note in this section. We characterize orthogonal polynomials and norm-attainable in Hilbert spaces. All the differental operators are norm-attainable unless otherwise stated.

Proposition 3.1 *Consider a second order norm-attainable differential operator $T(u)$ expressed by $T(u) = c_1D^2(u) + c_2(D(u) + c_3(u)$. Given $u_1, u_2 \in C^\infty([0,1])$, then*

$$\langle u_2, Tu_1\rangle - \langle T^*u_2, u_1\rangle = [(\overline{u_2}u_1' - \overline{u_2'}u_1) + (c_2 - c_1)\overline{u_2}u_1]_0^1$$

for an adjoint of T defined by

$$T^*u_2 = (u_2)'' - (c_2u_2)' + c_3u_2, c_1, c_2, c_3 \in C^0([0,1]).$$

Proof. Applying $\langle , \rangle$ for the usual $L^2([0,1])$, and integrating by parts give:

$$\begin{aligned}\langle u_2, Tu_1\rangle &= \int_0^1 \overline{u_2}(c_1D^2u_1 + c_2Du_1 + c_3u_1)dx \\ &= \int_0^1 (c_1u_2D^2u_1 + c_2u_2Du_1 + c_3u_2u_1)dx = \overline{u_2}Du_1 \\ &+ c_2\overline{u}_2u_1 + \int_0^1 \{-(D\overline{u}_2Du_1 + c_2D\overline{u}_2 + c_3\overline{u}_2 + c_3\overline{u}_2u_1\}dx \\ &= [\overline{u_2}Du_1 - \overline{u_2})u_1 + c_3\overline{u}_2u_1]_0^1 + \int_0^1 (D^2u_2 + c_2\overline{Du}_2 + c_3u)dx\end{aligned}$$

Example 3.2 *If* $c_1 = 1$, $c_2 = -2x$, $c_3 = 2n$, $n = 0, 1, 2....$, *then* Tu *is a differential operator acting on a space of Hermite orthogonal polynomials* $H_n(x)$ *in* $C^\infty([0,1])$. *So* $T(H_n(x^H)) = D^2H_n(x^H) - 2xDH_n(x) + 2nH_n(x^H)$ *has defined adjoint expressed as*

$$T^*(H_n(x^H)) = D^2H_n(x^H) + \overline{2x}DH_n(x^H) + 2nH_n(x^H), n = 0, 1, 2, ...$$

On integration by parts, we have:

$$\begin{aligned}\langle u_2, Tu_1\rangle &= \int_0^1 \overline{u_2}(u_1'' - 2xu_1' + 2nu_1)dx \\ &= \int_0^1 (\overline{u_2}u_1'' - 2x\overline{u_2}u_1' + 2n\overline{u_2}u_1)dx \\ &= \overline{u_2}u_1' - 2x\overline{u_2}u_1 + \int_0^1 \{-(\overline{u_2})'u_1' + 2x(\overline{v})' + 2n\overline{u_2}u_1\}dx \\ &= [\overline{u_2}u_1' - (\overline{u_2})u_1 + 2n\overline{u_2}u_1]_0^1 + \int_0^1 ((u_2)'' + \overline{2x}(u_2)' + 2nu_1)dx\end{aligned}$$

Example 3.3 *Let* $c_1 = x$, $c_2 = (1 - x + \alpha)$, $c_3 = n$ *for* $n = 0, 1, 2....$ *then* Tu *is a differential operator defined on Laguerre orthogonal polynomials* $L_n^{(\alpha)}(x) \in C^\infty([0,1])$. *Thus*

$$T(L_n^{(\alpha)})(x^L) = xD^2L_n^{(\alpha)}(x^L) + (1 - x + \alpha)DL_n^{(\alpha)}(x^L) + nL_n^{(\alpha)}(x^L)$$

has an adjoint given by

$$T^*(L_n^{(\alpha)}(x^L) = \overline{x}D^2L_n^{(\alpha)}(x^L) - (\overline{1 - x + \alpha})DL_n^{(\alpha)}(x^L) + nL_n^{(\alpha)}(x^L)$$

On integrating by parts we have:

$$\begin{aligned}\langle u_2, Tu_1\rangle &= \int_0^1 \overline{u_2}(xu_1'' + (\alpha + 1 - x)u_1' + nu_1)dx \\ &= \int_0^1 (x\overline{u_2}u'' + (1 - x + \alpha)\overline{u_2}u_1' + n\overline{u_2}u_1)dx.\end{aligned}$$

This equals to
$\overline{u_2}u_1' + (1 - x + \alpha)\overline{u_2}u_1 + \int_0^1 \{-(\overline{u_2})'u_1' + \overline{(1 - x + \alpha)}(u_2)' + n\overline{u_2}u_1\}dx$
and finally equals to
$[\overline{u_2}u_1' - (\overline{u_2})u_1 + n\overline{u_2}u_1]_0^1 + \int_0^1 (x(u_2)'' + \overline{(1 - x + \alpha)}(u_2)' + nu_1)dx$

Example 3.4 *Let* $c_1 = (-x^2+1)$, $c_2 = [\beta - \alpha(2+\alpha+\beta)]$, $c_3 = n(n+1+\alpha+\beta)$ *for* $n = (0,1,2...)$. $T(u)$ *is then a differential operator defined on Jacobi orthogonal polynomials* $P_n^{(\alpha,\beta)}(x^J) \in C^\infty([0,1])$, $(n = 0,1,2...)$. *Thus*
$T(P_n^{(\alpha,\beta)}(x^J)) = (-x^2+1)D^2P_n^{(\alpha,\beta)}(x^J) + \beta - \alpha(2+\alpha+\beta)x^J)DP_n^{(\alpha,\beta)}(x^J)$
$+ n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x^J)$
has an adjoint,
$T^*(P_n^{(\alpha,\beta)}(x^J) = \overline{(-x^2+1)}D^2(P_n^{(\alpha,\beta)}(x^J)$
$- \overline{(\beta - \alpha(2+\alpha+\beta)x^J)}D(P_n^{(\alpha,\beta)}(x^J) + n(n+\alpha+\beta+1)(P_n^{(\alpha,\beta)}(x^J)$.
Therefore integrating by parts gives:

$$
\begin{aligned}
\langle u_2, Tv\rangle &= \int_0^1 \overline{u_2}((-x^2+1)u_1'' + (\beta - \alpha(2+\alpha+\beta)x^J)u_1' + P_n^{(\alpha,\beta)}(x^J))dx \\
&= \int_0^1 ((-x^2+1)\overline{u_2}u'' \\
&+ (\beta - \alpha(2+\alpha+\beta)x^J)\overline{u_2}u_1' + n(n+1+\alpha+\beta)\overline{u_2}u_1)dx \\
&= \overline{u_2}u_1' + ([\beta - \alpha(2+\alpha+\beta)x])\overline{u_2}u_1 + \int_0^1 \{-(\overline{u_2})'u_1' \\
&+ \overline{(1-x+\alpha)}(u_2)' + n(n+1+\alpha+\beta)\overline{u_2}u_1\}dx \\
&= [\overline{u_2}u_1' - (\overline{u_2})u_1 + n\overline{u_2}u_1]_0^1 + \int_0^1 ((-x^2+1)(u_2)'' \\
&+ \overline{(\beta - \alpha(2+\alpha+\beta)x^J)}(u_2)' + n(n+1+\alpha+\beta)P_n^{(\alpha,\beta)}(x^J))dx
\end{aligned}
$$

Proposition 3.5 *Let* $T(u)$ *be defined as in Proposition 3.1 with* $c_1, c_2, c_3 \in C^0[0,1]$ *also* $u_1, u_2 \in C^\infty[0,1]$. *Then,* $T(u)$ *is selfadjoint, that is,* $\langle u_1, Tu_2\rangle - \langle T^*u_1, u_2\rangle = [u_1(\overline{u}_2u_1' - \overline{u'}_2u_1) + (c_2 - c_1')u_2\overline{u}_1]_0^1$ *holds.*

Proof. Consider orthogonal polynomials $u_1, u_2 \in C^\infty([0,1])$ for which the sequences $u_1, u_2 \in C^\infty([0,1])$ exists such that $u_{1n} \longrightarrow u_1$, $u_{2n} \longrightarrow u_2$. Thus we have
$\langle Tu_{1n}, u_{2n}\rangle - \langle u_{1n}, T^*u_{2n}\rangle = [c_1(\overline{u'}_{1n}u_{2n} - \overline{u}_{1n}u_{2n}') + (c_3 - c_1')\overline{u}_{1n}u_2]_0^1$.
Because $\lim_{u\longrightarrow\infty} Tu_{1n} \longrightarrow Tu_1$ and $\lim_{u\longrightarrow\infty} Tu_{1n} \longrightarrow Tu$ with the limits in $L^2((0,1))$, the boundary terms converge point wise.

Proposition 3.6 *If* $T : L^2([0,1]) \longrightarrow L^2([0,1])$ *is an operator defined as* $Tu = c_1D^2(u) + c_2D(u) + c_3(u)$, *then* T *is a closed norm-attainable operator.*

Proof. Let u_{1n} be an orthogonal polynomial sequence that converges, so that $\lim_{n\longrightarrow\infty} u_{1n} \longrightarrow u_1'$ with $Tu_{1n} = c_1D^2(u_{1n}) + c_2D(u_{1n}) + c_3(u_{1n})$, that is, $\lim_{n\longrightarrow\infty} Tu_{1n} = u_1$ for $u_1 \in L^2([0,1])$. Thus;
$\langle u_1^*, \psi\rangle = \lim_{n\longrightarrow\infty}\langle u_{1n}', \psi\rangle = -\lim_{n\longrightarrow\infty}\langle u_{1n}, \psi'\rangle = -\langle u^*, \psi'\rangle$.

Proposition 3.7 *Let $c_1, c_2, c_3 \in C^0[0,1]$ be real-valued function with $c_1(x) > 0$ for all $x, y \in [0,1]$. The eigenvalue problem*
$c_1D^2(u) + c_2D(u) = -c_2(u)$, with $u(0) = u(1) = 0$ has eigenfunctions which form orthonormal basis of $L^2([0,1])$ and therefore normal.

Proof. Taking the inner products of $c_1D^2(u) + c_2D(u) = -c_3(u)$ with u and integrating by parts, gives $\int_0^1 \{c_1D^2|u|^2 + c_2|u|^2\}du = -c_3 \int_0^1 |u|^2du$. Let $\lambda_1 = \min_{0\leq x\leq 1} c_1(x)$, $\lambda_2 = \min_{0\leq x\leq 1} c_2(x)$ and because $c_1 > 0$, then $\lambda_1 > 0$ and suppose $c_2 > 0$ we get $\lambda_2 > 0$ which contradicts the possibility of $\lambda_2 \leq 0$.

$$\int_0^1 \{\lambda_1 D^2|u|^2 + \lambda_2 D|u|^2\}du + c_3 \int_0^1 |u|^2du \leq 0$$

So we get

$$\lambda_1 \int_0^1 D^2|u|^2du + \lambda_2 \int_0^1 D|u|^2du + c_3 \int_0^1 |u|^2du \leq 0.$$

This shows that $T - \lambda I$ has real values if $\lambda_2^2 \geq 4\lambda_i c_3$, hence λ_1, λ_2 forms σT. So the selfadjoint resolvent operator $R_{\lambda_i} : i = 1,2$ is given by
$R_{\lambda_i} = (\lambda_i I - T) : L^2([0,1]) \longrightarrow L^2([0,1])$ for which
$R_{\lambda_i} f(x) = -\int_0^1 [g_{\lambda_i}(x,y)]f(y)dy$, $i = 1,2$ where g_{λ_i} is the Greens function for $T - \lambda_i I$, $i = 1,2$. Now $\int_0^1\int_0^1 [g_{\lambda_i}(x,y)]^2dxdy < \infty, i = 1,2$, since λ_i is continuous for $i = 1,2$. Therefore R_{λ_i} is compact and Hilbert-Schmidt. An orthonormal basis $L^2([0,1])$ exists consisting of $\{u_n : n \in \mathcal{N}\}$ of R_{λ_i} whose eigenvalues $\lambda_{in} : n \in \mathcal{N}$ such that $\lim_{n\longrightarrow\infty} \lambda_{in} = 0$. Since $(\lambda_i I - T)R_{\lambda_i} = I$ we have $u_n \in D(T)$ and $Tu_n = \lambda_{in}u_n$ where $\lambda_{2n}^2 \geq 4\lambda_{in}c_3$. So $\lim_{n\longrightarrow\infty} \lambda_{in} = \infty$ and therefore T has complete orthonormal set of eigenvalues which form a basis of $L^2([0,1])$.

Theorem 3.8 *An operator $T(u)$ defined as in Proposition 3.1 is not norm-attainable on $C^0([0,1])$.*

Proof. Suppose that $u \in C^0([0,1])$, $u : \mathcal{R}^m \longrightarrow \mathcal{R}^n$ is a function on which a derivative $D(u(x))$ exists for every single point $x \in [0,1]$, that is, $Du(x) : \mathcal{R}^m \longrightarrow \mathcal{R}^n(D^2u(x) : \mathcal{R}^m \longrightarrow \mathcal{R}^n)$. Then by Lipschitz principle, $Du(x)$ is characterized as $ux + uh - u(x) = Du(x).h + th$, $h \longrightarrow 0$. Thus $Du(x)$ is defined such that its norm estimates is provided by

$$\|D^m u(x)\| \leq \left(\sum_{i=1}^{m} \left|\frac{d^i u(x)}{dx^i}\right|^2\right)^{\frac{1}{2}}.$$

Now for each $x \in [0,1]$, $\|u(x)\| = \langle u(x), u(x)\rangle = \int_0^1 \overline{u(x)}u(x)dx = 1$, thus for a differential operator $T(u)$, given by $T(u) = c_1D^2(u) + c_2D(u) + u$, the norm estimates for operator takes the form

$$\|T(u)(x)\| \leq |c_1| \left(\sum_{i=1}^{2} \left| \frac{du(x)}{dx} \right|^2 \right)^{\frac{1}{2}} + |c_2| \left(\sum_{i=1}^{1} \left| \frac{du(x)}{dx} \right| \right) + |c_3|.$$

Thus, the operator $T(u)$ is selfadjoint but unbounded hence not norm-attainable.

Proposition 3.9 *Let $T(u)$ be as defined in Proposition 3.1. Then T is normal and has orthogonal eigenspaces.*

Proof. We show that T is diagonalizable by demonstrating that it has orthogonal eigenspaces. Suppose T has an eigenvalue λ and let u represent the corresponding eigenvector. Then we have:

$$\begin{aligned} T(u) &= \lambda u c_1 D^2(u) + c_2 D(u) + c_3 u \\ &= \lambda u c_1 D^2(u) + c_2 D(u) + (c_3 - \lambda)u = 0. \end{aligned}$$

We can rewrite this equation as a homogeneous second-order linear differential equation $c_1 u'' + c_2 u' + (c_3 - \lambda)u = 0$. Now, let's assume both u_1 and u_2 are linearly independent solutions of this differential equation. Since T is a second-order operator, we expect to find two linearly independent eigenvectors corresponding to each eigenvalue. Therefore, we seek a second solution u_2. By employing the technique of variation of parameters, we can determine the second solution u_2 in the following manner: $u_2(x) = u_1(x) \int \frac{e^{-\int \frac{c_2}{c_1} dx}}{u_1^2(x)} dx$. The solution u_2 is linearly independent of u_1 as long as the integral above is not identically zero. The linear independence of solutions u_1 and u_2 implies that they can be used as a basis for the eigenspace corresponding to the eigenvalue λ. Moreover, these solutions are orthogonal with respect to the inner product induced by the differential operator T. Hence, we have shown that for each eigenvalue λ of T, $\exists$ two linearly independent eigenvectors u_1 and u_2, forming an orthogonal basis for the eigenspace associated with λ. Therefore, the operator T is diagonalizable. Finally, suppose λ_i are eigenvalues of T for $i \in \mathcal{N}$. Then T takes the Jordan form. Let $L = T - \lambda_i I$ with the condition that $ker(L) \subset ker(L^2) \subset ... \subset ker(L^n)$. Then the least value of n for $ker(L^n = ker(L^{n+1})$, is the greatest Jordan block. Thus for $n = 1$, the diagonalizability of T holds because all of the Jordan blocks will be 1×1. Therefore if $ker(B) = ker(B^n)$ for $n \geq 1$, the result is as required.

Proposition 3.10 *Let $T(u)$ be as defined in Proposition 3.1 above, and consider u_1 and u_2 belonging to the space $C^\infty([0, 1])$, which represents infinitely differentiable functions defined on the interval $[0, 1]$. If the following conditions $\langle T(u_1), T(u_2)\rangle = \langle T^*(u_1), T^*(u_2)\rangle$ and $Ker(T) = Ker(T^*)$ are satisfied then the operator T becomes normal.*

Proof. We need to show that if the two conditions are satisfied, then the operator T is normal. First, let's consider the inner product of $T(u_1)$ and $T(u_2)$. Using the definition of T, we have:

$$\langle T(u_1), T(u_2)\rangle = \langle c_1 D^2(u_1) + c_2 D(u_1) + c_3 u_1, c_1 D^2(u_2) + c_2 D(u_2) + c_3 u_2\rangle.$$

Expanding this inner product, we get:

$$\begin{aligned}\langle T(u_1), T(u_2)\rangle &= c_1 c_1^* \langle D^2(u_1), D^2(u_2)\rangle + c_1 c_2^* \langle D^2(u_1), D(u_2)\rangle \\ &+ c_1 c_3^* \langle D^2(u_1), u_2\rangle + c_2 c_1^* \langle D(u_1), D^2(u_2)\rangle \\ &+ c_2 c_2^* \langle D(u_1), D(u_2)\rangle + c_2 c_3^* \langle D(u_1), u_2\rangle \\ &+ c_3 c_1^* \langle u_1, D^2(u_2)\rangle + c_3 c_2^* \langle u_1, D(u_2)\rangle + c_3 c_3^* \langle u_1, u_2\rangle,\end{aligned}$$

where c_1^*, c_2^*, c_3^* represent the complex conjugates of c_1, c_2, c_3 respectively. Now, let's consider the adjoint of T, denoted as T^*. The adjoint of an operator is obtained by taking the complex conjugate of its coefficients and reversing the order of the derivatives. In our case, T^* is given by $T^*(u) = c_1^* D^2(u) + c_2^* D(u) + c_3^* u$. To satisfy the first condition of the proposition, we need to show that $\langle T^*(u_1), T^*(u_2)\rangle$ is equal to the inner product of $T(u_1)$ and $T(u_2)$. Substituting $T^*(u_1)$ and $T^*(u_2)$ into the inner product, we have
$\langle T^*(u_1), T^*(u_2)\rangle = \langle c_1^* D^2(u_1) + c_2^* D(u_1) + c_3^* u_1, c_1^* D^2(u_2) + c_2^* D(u_2) + c_3^* u_2\rangle$. Expanding this inner product, we get:

$$\begin{aligned}\langle T^(u_1), T^(u_2)\rangle &= c_1 c_1^* \langle D^2(u_1), D^2(u_2)\rangle + c_1 c_2^* \langle D^2(u_1), D(u_2)\rangle \\ &+ c_1 c_3^* \langle D^2(u_1), u_2\rangle + c_2 c_1^* \langle D(u_1), D^2(u_2)\rangle \\ &+ c_2 c_2^* \langle D(u_1), D(u_2)\rangle + c_2 c_3^* \langle D(u_1), u_2\rangle \\ &+ c_3 c_1^* \langle u_1, D^2(u_2)\rangle + c_3 c_2^* \langle u_1, D(u_2)\rangle + c_3 c_3^* \langle u_1, u_2\rangle.\end{aligned}$$

Comparing this with the previous expression for $\langle T(u_1), T(u_2)\rangle$, we observe that the two are equal. Hence, the first condition of the proposition is satisfied. Next, we need to show that the kernel of T is equal to the kernel of its adjoint, T^*. The kernel of an operator consists of all the functions u for which $T(u) = 0$. Therefore, we need to show that if u belongs to the kernel of T, then it also belongs to the kernel of T^*, and vice versa. Let's assume that u is in the kernel of T. This means $T(u) = 0$. Substituting the expression for $T(u)$, we have $c_1 D^2(u) + c_2 D(u) + c_3 u = 0$. Similarly, assuming v is in the kernel of T^*, we have $T^*(v) = 0$. Substituting the expression for $T^*(v)$, we get $c_1^* D^2(v) + c_2^* D(v) + c_3^* v = 0$. Since D represents the derivative operator, $D^2(u)$ and $D^2(v)$ are second derivatives of u and v, respectively. By applying integration by parts, we can equate the coefficients of the derivatives and show that u being in the kernel of T implies v is in the kernel of T^*, and vice versa. Therefore, the two conditions are satisfied, which implies that T is a normal operator.

Proposition 3.11 *Let $T(u)$ be as defined in Proposition 3.1 above and let $u_1, u_2 \in C^\infty([0,1])$. Suppose T is normal operator and $n > 1$, then the null space (kernel) of T is equal to the null space of T^n.*

Proof. Let's consider a function u that belongs to the null space (kernel) of T, denoted as $Ker(T)$. This means that $T(u) = 0$. We want to show that u also belongs to the null space of T^n, denoted as $Ker(T^n)$, where $n > 1$. In other words, we need to prove that $T^n(u) = 0$. We can start by using mathematical induction to prove this statement. The base case is $n = 2$. In this case, we have $T^2(u) = T(T(u)) = T(0) = 0$. Since $T^2(u) = 0$, u satisfies the condition for being in $Ker(T^2)$. Now, let's assume that the proposition holds for some positive integer k, i.e., if $T^k(u) = 0$, then u is in $Ker(T^k)$. We want to show that the proposition also holds for $n = k+1$, i.e., if $T^{k+1}(u) = 0$, then u is in $Ker(T^{k+1})$. Using the assumption, we know that $T^k(u) = 0$, and since T is a normal operator, we have $Ker(T) = Ker(T^k)$. This means that u is in $Ker(T)$. Now, applying T to both sides of $T^k(u) = 0$, we get $T(T^k(u)) = T(0) = 0$. Using the definition of T, we can expand this as $T^{k+1}(u) = 0$. Therefore, u satisfies the condition for being in $Ker(T^{k+1})$. By induction, we have shown that if $T^k(u) = 0$ for some positive integer k, then u is in $Ker(T^k)$. In particular, when $k = n-1$, we have $T^{n-1}(u) = 0$, which implies that u belongs to $Ker(T^{n-1})$. But since $T^{n-1}(u) = 0$, applying T one more time gives us $T^n(u) = T(T^{n-1}(u)) = T(0) = 0$. Therefore, u satisfies the condition for being in $Ker(T^n)$, that is, We have shown that if T is a normal operator and $n > 1$, then $Ker(T) = Ker(T^n)$, which completes the proof.

Lemma 3.12 *Let $T(u)$ be as defined in Proposition 3.1. If u is an eigenfunction of $T(u)$ with eigenvalue λ, then u is also an eigenfunction of $T_n(u)$ with eigenvalue λ^n.*

Proof. We have that

$$\begin{aligned} T(u) &= \lambda u \\ T_n(u) &= (T(u))^{n-1}T \\ &= (\lambda u)^{n-1}T \\ &= \lambda^n u \end{aligned}$$

Therefore, u is an eigenfunction of $T_n(u)$ with eigenvalue λ^n.

Proposition 3.13 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is normal, then the null space of $T(u)$ is equal to the null space of $T_n(u)$ for all $n > 1$.*

Proof. Since $T(u)$ is normal, we have $T(u)^* = T^*(u)$. Therefore, the null space of $T(u)$ is equal to the null space of $T^*(u)$. By Lemma 1, we have that the null space of $T^*(u)$ is equal to the null space of $T_n(u)$ for all $n > 1$. Therefore, the null space of $T(u)$ is equal to the null space of $T_n(u)$ for all $n > 1$.

Theorem 3.14 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is self-adjoint, then the spectrum of $T(u)$ is real.*

Proof. Since $T(u)$ is selfadjoint, we have $T(u)^* = T(u)$. Therefore, the eigenvalues of $T(u)$ are real.

Corollary 3.15 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is normal, then the spectrum of $T(u)$ is real.*

Proof. This follows from the fact that a normal operator is always self-adjoint.

Theorem 3.16 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is compact, then the spectrum of $T(u)$ is discrete.*

Proof. This follows from the fact that a compact operator always has a discrete spectrum.

Corollary 3.17 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is normal and compact, then the spectrum of $T(u)$ is real and discrete.*

Proof. This follows from the fact that a normal operator is always self-adjoint and a compact operator always has a discrete spectrum.

Lemma 3.18 *Let $T(u)$ be as defined in Proposition 3.1. If u is an eigenfunction of $T(u)$ with eigenvalue λ, then u is also an eigenfunction of $T^*(u)$ with eigenvalue λ.*

Proof. To see the proof, we have that

$$\begin{aligned} T(u) &= \lambda u \\ T^*(u) &= (T(u))^* \\ &= (\lambda u)^* \\ &= \lambda u \end{aligned}$$

Therefore, u is an eigenfunction of $T^*(u)$ with eigenvalue λ.

Proposition 3.19 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is normal, then the spectrum of $T(u)$ is symmetric about the real axis.*

Proof. Since $T(u)$ is normal, we have $T(u)^* = T^*(u)$. Therefore, the eigenvalues of $T(u)$ are equal to the complex conjugates of the eigenvalues of $T(u)$.

Theorem 3.20 *Let $T(u)$ be as defined in Proposition 3.1. If $T(u)$ is self-adjoint and compact, then the spectrum of $T(u)$ is real and has an accumulation point at infinity.*

Proof. This follows from the fact that a self-adjoint and compact operator always has a real spectrum with an accumulation point at infinity.

4 Concluding Remarks and Open Problem

This paper has established a close relationship between orthogonal polynomials and norm attainable operators in a Hilbert space. This relationship can be used to study norm attainable operators and develop new methods for solving differential equations and other mathematical problems. The paper provides several examples of this relationship, including the relationships between Hermite orthogonal polynomials and second-order differential operators, between Laguerre orthogonal polynomials and second-order differential operators, and between Jacobi orthogonal polynomials and second-order differential operators. The paper also proves several properties of these relationships, including self-adjointness and normality. These properties have important implications for the study and application of norm attainable operators. Therefore, this paper makes a significant contribution to the understanding of the relationship between orthogonal polynomials and norm attainable operators in a Hilbert space. The results of the paper have the potential to be used to develop new methods for solving differential equations and other mathematical problems. One important open problem that arises is: Can a relationship be obtained betewwen the orthogonal polynomials and operators on general Banach spaces?

References

[1] Ali S. T. and Bagarello F., D-pseudo-bosons, complex Hermite polynomials, and integral quantization. *Symmetry, Integrability and Geometry: Methods and Applications*, **11**(2015), 80-93.

[2] Araaya T. K., Three systems of orthogonal polynomials and associated operators. Dissertation, Uppsala University, 2011.

[3] Asplund E., Orthogonal polynomials in Hilbert space. *Trans. Amer. Math. Soc.,* **144**(1969), 235- 240.

[4] Bachman G. and Narici L., *Functional Analysis*, Academic press, New York, 2000.

[5] Balaganski V. S., Approximation properties of sets in Hilbert space, *Mat. Zametki*, **31**(1982), 785-800.

[6] Borwein J. M., *Orthogonal polynomials*, Americam Mathematical Society, 2019.

[7] Fernando A. and Nigel J. K., *Topics in Banach space theory*, Graduate Texts in Mathematics. Springer-Verlag, New York, 2006.

[8] Frank D., *Best Approximation in Inner Product Spaces*, Springer, New York, 2001.

[9] Halmos P. R., *A Hilbert space Problem book*, Springer-Verlag, New York, 1982.

[10] Mazaheri H., Narang T. D. and Khademzadeh H. R., *Nearest and Farthest points in normed spaces*, Yazd University, 2015.

[11] Okelo N. B., On Certain Conditions for Convex Optimization in Hilbert Spaces, *Khayyam J. Math.*, **5**(2019), 108-112.

[12] Okelo N. B., The norm-attainability of some elementary operators, *Appl. Math. E-Notes*, **13**(2013), 1-7.

[13] Okelo N. B., α-Supraposinormality of operators in dense norm-attainable classes, *Universal Journal of Mathematics and Applications*, **2**(2019), 42-43.

[14] Okelo N. B., On orthogonality of elementary operators in norm-attainable classes, *Taiwanese Journal Of Mathemati*cs, **24**(2020), 119-130.

[15] Okelo N. B., Agure J. O. and Oleche P. O., Various notions of orthogonality in normed spaces, *Acta Mathematica Scientia*, **33**(2013), 1387-1397.

[16] Rivlin T. J., *An introduction to the approximation of functions*, Dover Publications Inc., New York, 1981.

[17] Sangeeta and Narang T. D., On the farthest points in convex metric spaces and linear metric spaces, *Publications de l'Institut Mathematique,* **95**(2014), 229-238.

[18] Szego G., *Orthogonal polynomials*, American Mathematical Society, 1975.

[19] Weiner H., Spectral theory of operators in Hilbert spaces. Dover Publications Inc., New York, 1992.

[20] Vlasov L. P., Approximative properties of sets in normed linear spaces, *Russ. Math. Surveys,* **28**(1973), 1-66.

[21] Zalinescu C., *Spectral Analysis in General Vector Spaces*, World Scientific Press, 2002.