Int. J. Open Problems Compt. Math., Vol. 15, No. 4, December 2022
Print ISSN: 1998-6262, Online ISSN: 2079-0376

On Fixed Point Properties for Nonexpansive Norm-attainable Mappings in Fuzzy Skorohod Spaces

Benard Okelo

Department of Pure and Applied Mathematics,
Jaramogi Oginga Odinga University of Science and Technology,
Box 210-40601, Bondo-Kenya.
e-mail: benard@aims.ac.za

Received 29 August 2022; Accepted 8 December 2022

Abstract

The intricate nature of the underlying structures in Skorohod spaces presents a complex problem when studying fixed point theorems. This paper presents certain fixed point properties for nonexpansive mappings in fuzzy Skorohod spaces. We show that for a Fuzzy Skorohod Space $\mathcal{O}_{\mathcal{SK}}$, if $z \in \mathcal{O}_{\mathcal{SK}}$ is an interior point then, z is a fixed point if $\mathcal{F}$ is norm-attainable.

Keywords: *Fixed point, Fuzziness, Skorohod space, Properties, Uniqueness*
2010 Mathematics Subject Classification: 47H10.

1 Introduction

Studies on fuzziness remains interesting after the onset of fuzzy set (FS) theory by [18]. Characterizing properties of mappings in FS has been considered by many authors particularly in different spaces and algebras [19] and [20]. Fixed point theorems have also been considered in different aspects and in different spaces [14]. Among all these studies, the fixed point problem remains unresolved [13]. The fundamental question that has been given consideration for a very long time particularly in different special case is: Can one determine a fixed point property that applies to all spaces in general? This paper therefore considers a fuzzy Skorohod space (FSS) which is complete and norm-attainable (see [1]-[4] and thee references therein). The intricate nature of the underlying structures in Skorohod spaces presents a complex problem when studying fixed point theorems [2]. Norm-attainability as a property of mappings have also been considered in different algebraic settings [17]. For instance, conditions under which mappings in Hilbert space s become norm-attainable has been extensively discussed in [5]-[16]. This paper presents certain fixed point properties for nonexpansive mappings in fuzzy Skorohod spaces [3]. We derive the properties with the fuzziness aspect in mind when the FSS is complete and norm-attainable. The paper is organized as follows. In section one we provide a mathematical background to the study, in section 2 we give the preliminaries which include the basic concepts and terminologies which are useful in the sequel. In section 3 we give the main results and their discussions. Lastly we pose an open problem to be considered in future research.

2 Preliminaries

For a better understanding of this work, we provide certain basic concepts and new terminologies which are instrumental to this study. Let $\mathcal{L}$ be a complete FSS and $\mathcal{J} : \mathcal{L} \to \mathcal{L}$ be a mapping. A point $x \in \mathcal{L}$ is called a fixed point of $\mathcal{J}$ if $\mathcal{J}(x) = x$. The set of all fixed points of $\mathcal{J}$ is denoted by $F(\mathcal{J})$, that is $F(\mathcal{J}) = \{x \in H : \mathcal{J}(x) = x\}$, where H is a Banach space. It is known that $F(\mathcal{J})$ is closed and convex, for more details, see [1] for details.

Definition 2.1 *([15], Definition 2.1) A Skorokhod space is a family of functions $x : [0,T] \to \mathcal{R}^1$, which are right-continuous at every $t \in [0,T)$ and admit left-limits at every $t \in (0,T]$. It is naturally equipped with the sup-norm $\|x\|_\infty = \sup_{t\in[0,T]} |x(t)|$.*

Definition 2.2 *([19], Definition 1) Let X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs: $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : \ x \in X\}$. $\mu_{\tilde{A}}(x)$ is called the membership function (generalized characteristic function) which maps X to the membership space M. Its range is the subset of nonnegative real numbers whose supremum is finite. For $\sup \mu_{\tilde{A}}(x) = 1$ we have a normalized fuzzy set.*

Remark 2.3 *In Definition 2.2, the membership function of the fuzzy set is a crisp (real-valued) function. Zadeh [18] also defined fuzzy sets in which the membership functions themselves are fuzzy sets.*

Definition 2.4 *A fuzzy set μ on $\mathcal{R}$ is said to be a fuzzy number if the following properties are verified:*

(i). There exists $x_0 \in \mathcal{R}$ *such that* $\mu(x_0) = 1$,

(ii). μ *is an upper semi-continuous function,*

(iii). $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, *for all* $x, y \in \mathcal{R}$ *and* $\lambda \in [0,1]$, *and*

(iv). $[\mu]^0$ *is compact.*

Definition 2.5 *([20], Definition* 3.2*) A type* m *fuzzy set is a fuzzy set whose membership values are the type* $m-1, m > 1$, *fuzzy sets on* $[0,1]$.

Remark 2.6 *For operations on fuzzy set see [19] and the references therein.*

Definition 2.7 *([16]) Let* $T : \mathcal{M} \to \mathcal{M}$ *be a mapping. We say that* T *is:*

(i). $\eta-$*strongly monotone, if there exists a constant* $\eta > 0$ *such that*

$$\langle Tx - Ty, x - y \rangle \geq \eta \|x - y\|, \forall x, y \in H,$$

(ii). Contractive if

$$\|Tx - Tz\| \leq k \|x - z\|, \forall x, z \in H, \tag{1}$$

where $k \in (0,1)$.

(iii). Nonexpansive mapping if Equation 1 reduces to the following equation as $k = 1$. $\|Tx - Tz\| \leq \|x - z\|, \forall x, z \in H$.

(iv). Asymptotically nonexpansive if

$$\|T^n x - T^n z\| \leq k_n \|x - z\|, \forall n \geq 1 \; x, z \in H, where \;\; k_n \subset [1, \infty).$$

(v). Totally asymptotically nonexpansive

$$\|T^n x - T^n z\|^2 \leq \|x - z\|^2 + \upsilon_n \eta(\|x - z\|) + \mu_n, \forall n \geq 1.$$

We adopt the following notations: Fuzzy Skorohod space by FSS; Nonexpansive norm-attainable mapping by NNAM; Lower Semi-Continuous function by LSC and Upper Semi-Continuous function by USC.

3 Main results

Equipped with the preliminaries, we are at a position to present our results in this section. We begin with some auxiliary results that are useful in proving the main result.

Proposition 3.1 *Let $\mathcal{O}_{\mathcal{SK}}$ be a FSS, and $\mathcal{F} : \mathcal{O}_{\mathcal{SK}} \to \mathcal{O}_{\mathcal{SK}}$ a NNAM. Let ζ_n be a monotone sequence in $\mathcal{O}_{\mathcal{SK}}$. If $z \in \mathcal{O}_{\mathcal{SK}}$ is a central point then, z is a fixed point.*

Proof. Suppose that $\mathcal{O}_{\mathcal{SK}}$ is complete. Then every Cauchy sequence in $\mathcal{O}_{\mathcal{SK}}$ is convergent. This implies that ζ_n is convergent to z. Consequently, $\mathcal{F}(\zeta_n)$ converges to $\mathcal{F}(z)$. But from Banach's fixed point theorem, $\mathcal{F}(z) = z$. The rest of the proof follows trivially from the proof of convergence conditions in [14]. This completes the proof.

Lemma 3.2 *Let $\mathcal{O}_{\mathcal{SK}}$ be a FSS, and $\mathcal{F} : \mathcal{O}_{\mathcal{SK}} \to \mathcal{O}_{\mathcal{SK}}$ a NNAM which is a LSC with a finite lower bound. Let ζ_n be a monotone sequence in $\mathcal{O}_{\mathcal{SK}}$. If $z \in \mathcal{O}_{\mathcal{SK}}$ is a central point then, z is a fixed point if $\mathcal{F}$ is norm-attainable.*

Proof. From Proposition 3.1, suppose that $\mathcal{O}_{\mathcal{SK}}$ is separable and admissible. Then every Cauchy sequence in $\mathcal{O}_{\mathcal{SK}}$ is convergent. This implies that ζ_n is convergent to z. Consequently, $\mathcal{F}(\zeta_n)$ converges to $\mathcal{F}(z)$. But from Banach's fixed point theorem, $\mathcal{F}(z) = z$. Moreover, it is known that $\mathcal{F}$ is norm-attainable if there exists a unit vector $z \in \mathcal{O}_{\mathcal{SK}}$ such that $\|\mathcal{F}(z)\| = \|\mathcal{F}\|$. By Weierstrass totality principle [1] of norm-attainability [7] and by Hahn-Banach Theorem the assertion suffices. This completes the proof.

At this point we state our main result in the theorem below.

Theorem 3.3 *Let $\mathcal{O}_{\mathcal{SK}}$ be a FSS, and $\mathcal{F} : \mathcal{O}_{\mathcal{SK}} \to \mathcal{O}_{\mathcal{SK}}$ a NNAM which is a USC with a finite upper bound. Let ζ_n be a monotone sequence in $\mathcal{O}_{\mathcal{SK}}$. If $z \in \mathcal{O}_{\mathcal{SK}}$ is an interior point then, then z is a fixed point if $\mathcal{F}$ is norm-attainable.*

Proof. The proof of this theorem follows analogously from the proof of Lemma 3.2.

Corollary 3.4 *Let $\mathcal{O}_{\mathcal{SK}}$ be a closed FSS. If $\mathcal{F} : \mathcal{O}_{\mathcal{SK}} \to \mathcal{O}_{\mathcal{SK}}$ a NNAM which is a closed valued β-contraction such that $\mathcal{F}$ is densely compact on $\mathcal{O}_{\mathcal{SK}}$, then for each $\epsilon > 0$ with $\lambda > 1$ $\mathcal{F}$ has a fixed point.*

Proof. By Theorem 3.3 and definition of Skorohod spaces, the proof of this corollary follows analogously from the proof of Lemma 3.2 and the fact that the convergence follows the norm topology criterion..

Remark 3.5 *The result of Corollary 3.4 can be extended to other types of nonexpansive mappings as listed in Definition 3.4.*

4 Open Problem

In this work, we have addressed the intricate nature of the underlying structures in Skorohod spaces which presents a complex problem when studying fixed point theorems. We have given certain fixed point properties for non-expansive mappings in fuzzy Skorohod spaces. We have shown that that for a Fuzzy Skorohod Space $\mathcal{O}_{\mathcal{SK}}$, if $z \in \mathcal{O}_{\mathcal{SK}}$ is an interior point then, then z is a fixed point if $\mathcal{F}$ is norm-attainable. We pose an open problem for future research as below:

Does there exist a fixed point theorem that holds for an n-dimensional separable and complete FSS in general?

References

[1] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, New York, 1967.

[2] J. L. Kelley, *General Topology*, Springer Verlag, New York, 1975.

[3] B. Ogola, N. B. Okelo, and O. Ongati Certain Notions of Continuity in Bitopological Spaces, Int. J. Open Problems Compt. Math., **14**(4), (2021), 48–67.

[4] N. B. Okelo, *On Convex Optimization in Hilbert Spaces,* Maltepe J. Math., **2**(2019), 89–95.

[5] N. B. Okelo, J. O. Agure and D. O. Ambogo, *Norms of elementary operators and characterization of norm-attainable operators*, Int. J. Math. Anal., **24** (2010), 1197–1204.

[6] N. B. Okelo, *The norm-attainability of some elementary operators*, Appl. Math. E-Notes, **13** (2013), 1–7.

[7] N. B. Okelo, *α-Supraposinormality of operators in dense norm-attainable classes*, Universal Journal of Mathematics and Applications, **2** (2019), 42–43.

[8] N. B. Okelo, *On orthogonality of elementary operators in norm-attainable classes*, Taiwanese Journal of Mathematics, **24** (2020), 119–130.

[9] N. B. Okelo, M. O. Okongo and S. A. Nyakiti, *On projective tensor norm and norm-attainable α-derivations*, Int. J. Contemp. Math. Sciences, **5** (2010), 1969–1975.

[10] N. B. Okelo, J. O. Agure and P. O. Oleche, *Various notions of orthogonality in normed spaces*, Acta Mathematica Scientia, **33** (5) (2013), 1387–1397.

[11] N. B. Okelo, *On Norm-attainable operators in Banach Spaces,* J. Function Spaces, **1**(2020), 1–6.

[12] N. B. Okelo, *On certain conditions for convex optimization in Hilbert Spaces,* Khayyam J. Math., **5**(2019), 108–112.

[13] N. B. Okelo, Fixed points approximation for nonexpansive operators in Hilbert spaces, Int. J. Open Problems Compt. Math., **14**(1), (2021), 1–5.

[14] N. B. Okelo, Maximal Chain Properties for Mappings in Bitopological Spaces, Int. J. Open Problems Compt. Math., **15**(1), (2022), 34–38.

[15] N. B. Okelo and A. Onyango, Characterization of Topological Fuzzy Sets in Hausdorff Spaces, Trans. Fuzzy Sets Syst., **1**(2), (2022), 32-36.

[16] V. Bansaye and F. Simatos, *On the scaling limits of Galton–Watson processes in varying environment,* Electron. J. Probab. **20**(75), (2015), 1–36.

[17] S. Suantai, Y. Shehu and P. Cholamjiak, *Nonlinear iterative methods for solving the split common null point problem in Banach spaces,* Optimization Methods and Software, **34**(2019), 853–874.

[18] L. A. Zadeh, *Information and control,* Fuzzy sets **8**(1965), 338–353.

[19] H. J. Zimmermann, *Fuzzy set theory,* Comp. Stat., **2**(2010), 317-332.

[20] H. J. Zimmermann, *Fuzzy set theory and its applications,* 4th Ed., Springer Dordrechtg, New York, 2001.