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On Supraposinormality of Operators in Norm-Attainable Classes

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Abstract

In this paper, we characterize various notions of posinormality of operators in norm-attainable classes. We give a detailed theory on posinormality, coposinormality and supraposinormality of operators in norm-attainable classes. We also give some open problems in general Banach space setting for the supraposinormal operators.

Keywords: *Supraposinormality, Operator, Norm-attainable class.*
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1 Introduction

Hilbert spaces are key in several studies in mathematics and their advancements both trivial and non-trivial cases in mathematics and quantum theory. Brown [2] introduced a class of operators which obeys the condition that T^*T commutes with T. These operators are referred to as quasinormal operators. When computed routinely, they indicate that if T is quasinormal and $\lambda \neq 0$, then λT is quasinormal, but the translate $T + \lambda$ can be quasinormal only if T

is normal. Building on the same introduction, Brown [2] started by analyzing his result of which gave him a platform of giving the characteristics of a subclass that is very large of $B(H)$, set of operators that are linear and bounded $T : H \longrightarrow H$ on a space called Hilbert H. The study made it easier to search the different types of other operators so that it could be possible to come up with other different operators. Hence, in so doing, the study had to carry out some references such as $Z^*Z - ZZ^*$ as a self-commutant of Z, written as $[Z^*, Z]$. This meant that a self-adjointing operator J is greater than zero, that is, $\langle Jf, f\rangle \geq 0$ termed as positive if for all $f \in H$ and an operator Z is said to be normal but under one condition that $[Z^*, Z] = 0$. This was very interesting indeed since one discovery of an operator, leads to the discovery of another positive operator. Rhaly [27] studied hypornormal operators in relation to posinormality where the results recorded proved that hyponormal operators are necessarily posinormal. Further on the same, the author extended the study to coposinormality of which the results recorded, proved that hyponormal operators are not necessarily coposinormal. On studying the conjecture of hyponormality for a ordered positive integer Cesáro matrix, [28] studied the Cesáro matrix of order three and four. The results were given by employing positive normality, which employed interrupters that are diagonal and techniques that involves computations of elements from calculus, showed that order three and four matrices of Cesáro are co-positive normal. Using these results, the conjecture for matrices of order greater than four of Cesáro with positive integer were also studied. Daoxing [6] studied hyponormal operators in relation to rank one self-commutators where the analytical theory was developed for the pure hyponormal operators with self-commutators with rank one were covered and spectra which are connected finitely and closely bounded domains with analytic boundaries. The analytic model of class was studied alongside the two kernel $S\langle ., .\rangle$ and $E\langle ., .\rangle$, the double model on spaces of Hilbert for analytic functions of the resolvent sets and diagonalizing their adjoint on some spaces of Hilbert that are kernel. Further, the function of Pincus principal of hyponormal operator on some special case was also studied. It was found that if H is a rank one pure hyponormal operator with a self-commutatant and a function of rotational invariant Pincus principal, then, either $H = bU_{e+}$, where b is a non-zero complex number and U_{e+} is unilateral shift with multiplicity one. If H is cyclic but not invertible, or $\sigma(H) = \zeta : m \leq |\zeta| \leq n$, for $0 < m < n < \infty$ and $f(\zeta) = 1$ for $\zeta \in \sigma(H)$, then, H is cyclic invertible. Senthilkumar and Sherin [36] studied and characterized the composition of operators giving emphasis on the composition of weighted operators of operators that are normal on L^2 space and (α, β) on general weighted Hardy Space. By taking (X, Σ, λ) as a finite sigma space with a measure and D be a transformation from X into itself whose measure is not singular, then the composition transformation C, on $L^2(\lambda)$ induced D which was given by $Ck = kD$ for every

k in $L^2(\lambda)$. Since C was bounded, then it was called a composition operator on $L^2(\lambda)$. It is known that D induces a bounded composition operator C on ,$L^2(\lambda)$ under the condition that the measure λD^{-1} is continuously absolute when the measures λ and k are considered, is essentially bounded where k is a measure derivation of λD^{-1} when consideration is made to λ, which was referred to as Rado-Nikodym. The derivative measure $\lambda(D^N)^{-1}$ in reference with λ was represented by k^N, where D^N is a composition of D, n times. Every function k whose value is complex with essential bounding, produces an operator M_f on $L^2(\lambda)$ that is bounded, which is given by $M_k k = kk$, then it is called $C^*C = M_k$ for every $f \in L^2(\lambda)$. A weighted composition operator W (w.c.o) induced by D is a linearly transforming the values of a complex set Σ with a function that is measurable k of the form $Wk = wkD$, with w as a value with complex σ such that when $w = 1$, then, it suffices that W is an operator with composition. Rhoades and Rhaly [31] studied posinormality operators and extended it to the factorable matrices with a constant main diagonal.In the study, sufficient conditions for positive normal factorable matrices with a main diagonal constant to be hyponormal, were given which some Toeplitz and non-Toeplitz matrices satisfy. Various examples were given which were used by [31] to obtain a result that is more general on the same. Taking $R = R(\{a_k\}, \{c_l\})$ to be a factorable matrices with P a diagonal interrupter, assuming that $a_k, c_l \neq 0$ for all k, l to obtain P, we engage the following matrix $Y = [b_{kl}]$ where by Y is bounded on ℓ^2 and both $\{a_n\}$ and $\{a_n/c_n\}$ are decreasingly positive sequence that is convergent to 0, then R is posinormal since $R = BR$. The same R is R-hyponormal if for all f in ℓ^2, $\langle (R^*R - RR^*)f, f\rangle = \langle (R^*R-(R^*B^*)(BR)f, f\rangle = \langle (I-B^*B)Rf, Rf\rangle \geq 0$. Consequently, it was concluded that R was to be hypornormal when $Q = I - P \geq 0$, where $P = B^*B$ where the range of R is composed of all the $e_n's$ extracted from a basis that is standard and orthonormal forℓ^2. Rhaly [30] extended the study of hyponormality and dominant operators in relation to lower factorable triangular matrix Y, acting as a bounded linear operator on ℓ^2. Conditions that sufficiently qualify the matrices of lower triangle which are factorable to be hyponormal and dominant on ℓ^2 were given. Further, it was shown on a space Hilbert H, that $B(H)$ represents the set of operators that are linear and bounded, then $E \in B(H)$ is hyponormal if $E^*E - EE^* \geq 0$. Further, the operator is called dominant for all λ in the spectrum of A, if the $Ran(E - \lambda) \subset Ran(E - \lambda)^*$. In conclusion, the author proved that hyponormal operators are necessarily dominant operators. Amelia [**??**] studied posinormality in relation to hyponormality. The equation $AA^* \leq \lambda^2 A^*A$, is posinormal if and only if $\lambda = c$ rendering the equation to be $AA^* \leq c^2 A^*A$. Agure, Okelo and Oleche [22] studied elementary operators and used the concepts learnt to give an analogy on norm-attainability of elementary operators and derivations. In these study, conditions that are sufficient for norm-attainability to hold were given. Okelo [23] studied norm-attainability in specific operators that are elementary and gave an analogy on when operators that are elementary become orthogonal. This was possible due to the fact that characterizations of these operators that are elementary have been done by a good number of mathematicians various diversities which gave an opportunity for the study of orthogonality which had not been studied. to be. First, the necessary and sufficient conditions of norm-attainability were given which were used to give the results on implementation of operators that are elementary on when the range and the kernel of these operators are orthogonal to operators in norm-attainable and their classes. Sadiq [33] studied quasi-positive normal operators focusing on the following sets of operators; normal, hyponormal, M-hyponormal, dominant and positive normal. Kostov [8] introduced an extension of group of hyponormal and positive normal operators naturally where a family of eigen distributions, unitary invariants, and model function were constructed. A property of operators from hyponormal class that is well known for T is $Im \subset [T^*] = \{T^*K : K \in B(H)\}$ which is an ideal and right in algebra $B(H)$, which T^* generates. These group of operators in hyponormal class extended to the positive normal operators that were introduced and researched by Rhaly [26]. On studying the functional models for posinormal operators, a class of (p) positive normal operators that are polynimial of $p = \Sigma_{k=1}^{n} a_k z^k$ for which there is a polynomial with a term that is constant and zero where $P(T) \in [T^*]$. Gopal [5] studied supraposinormality where the concept was used to give an analogy of generalized quasi-posinormal operators by describing some properties for the operators A on a space of Hilbert H satisfying $(E^*E)^k \leq c^2 E^{*k} E^k$ for some $c > 0, k \geq 2$ and also presented some characterizations for the composition operators and the weighted composition operators on the Hilbert L^2 to be of this type. Further, posi-(M, k) operators were studied and presented some properties along with certain equivalent conditions for an operator to be posi-(M, k). Strict inclusion of (M, k) class of operators in posi-(M, k) was also brought into picture. Moreover, the derivatives of conditions for composition and weighted composition operators on $L^2(\Omega, \mu, A)$ to be in Posi-(M, k) class. Duggal [7] defined a dominant operator that is; for a real number, M_λ, $\|(A-\lambda)^*x\| \leq M_\lambda \|(A-\lambda)x\| \; \forall \; x \in H$ holds. In any case, if M exists as a constant and $M_\lambda \leq M \; \forall \; \lambda$, then the operator A that is dominant is implied to be M-hyponormal. Further, a power bounded operator was discussed where by an operator A, sup $\|A^n\| \leq M$, holds with M being a number that is positive, then we take over the supremum for all numbers n that are natural. On studying dominant operators [7] gave consideration of linear transformation that are bounded on a space of Hilbert H "into itself or into another Hilbert space H_1", using the results of the theorem of Putnam and Fuglede which states, whenever A and B^* are operators that are normal, then $AX = XB$ holds for operator X. Equally $A^*X = XB^*$ simultaneously. Further, Moore et. al. [16] showed that since $AX = XB$ is satisfied for A

and B^* operators that are M-hyponormal for X, then $A^* = XB^*$ for injective operator X whose range is dense. This observation was employed to prove that Fuglede-Putnam operator holds for an operator A which is dominant and an operator B^* which is M-hyponormal . Consequently for $AX = XB$ on an operator A which is dominant and an operator B^* which is M-hyponormal with the completion of orthogonality for hermitian operator X with a kernel and the restriction of B, then $AX = XB^*$. Nickolov [18] studied posinormality and used it to give a conjecture of totally p-posinormality operators whereby an operator is p-posinormal if for $T \in B(H)$ and a polynomial P, $\|\overline{P}(T_z^*)h\| \leq M(z)\|T_z h\|$ exists for every $h \in H$. Also by the compacts of $\mathbf{C}$ with an operator $M(z)$ which is bounded, then $T_z = T - zI$. Further, it was proved that every operator that is totally positive normal is a sub-scalar, such that, it is restricted to a general operator with scalar and to a subspace that is invariant together with the presentation of corollaries that are significant on the property of Bishop β and availability of sub-spaces that are invariant. Lee [11] carried a study on the powers of P-posinormal operators where the analogy for an operator to be p-posinormal was given where for if $E \in B$, then $(EE^*)^g \leq \mu(E^*E)^g$ for $\mu > 1$ is g-posinormal. Further, it was proved that whenever E is p-positive normal, then E^n is g-positive normal as well for an integer n that is positive. It was shown moreover that, if $E = U|E|$ is g-positive normal for $0 < g < 1$, then by the transformation of Aluthge, $\overline{E} = |E|^{\frac{1}{2}}U|E|^{\frac{1}{2}}$ is $(g + \frac{1}{2})$-positive normal. Rhaly [29] studied hyponormality in relation to the weighted mean matrix whose sequence has weight and coefficients that are positive, linear, the results given showed that it was a posinormal operator on ℓ^2. Further, the weighted mean matrix proved to be co-positive normal, such that, there is same space that is null and range for it and its adjoint. In conclusion, it was shown that results of posinormality obtained showed that the mean matrix with weight that is generated by odd integers that are positive with sequence is hyponormal, a case that is more general concerning this conjecture. Mecheri [14] carried out a research on the generalized theorem of Weyl for positive normal operators where results showed that generalized theorem for Weyl holds for $f(E)$ if E is conditionally totally positive normal or totally positive normal was provided, such that there is a function $f_\circ$ that is analytic in a neighborhood that is open for $\sigma(E)$. Posinormality was defined equivalently to the one given by Rhaly [3]. Hence, $T_\circ$ as an operator is positive normal whenever a coisometry V^* and a non-negative bounded Hilbert space operator P, then $T = T^*PV^*$. It was realized that a large class of posinormal operator contains other classes such as hyponormal operators, M- hyponormal operators and dominant operators. Itoh [13] commented on the existence of hyponormal operators under the condition that $[T^*, T]$ is positive, implying that if T^* is hypornormal, then it is depicted that T is referred to as cohyponormal. That one was not enough, from hyponormality and cohyponormality, another operator developed such that in the case of T being both hyponormal or cohyponormal, then it is referred to as seminormal. If T is by any chance restricted in an operator that is normal to a sub-space that is invariant, then it is known as sub-normal. Rhaly [25] showed that all unilateral weighted shifts that are injective, are supra-posinormal. Kostov and Todorov [8] introduced operators called polynormially positive normal operators, which is a natural extension of hyponormality and posinormality operators. Further, they constructed a family generation of eigen distributions, invariants that are unitary and developed a model function for this class. Elaborately, it was discovered that carrying an extension on operators that are hyponormal to operators possessing the property $\Im T \subset \Im T^*$, positive normal operators are obtained. The class of polynormially posinormal operators comprises of all nilpotent operators, finitely dimensional operators together with all posinormal operators, which renders it larger than the class of M- hyponormal operators. Senthil [34] investigated continuity of spectrum for a (p, k)-quasipositive normal operator alongside (p, k)-quasi-hyponormal operator. Further, the (p, k)-quasipositive normal operator were shown to be a pole for the adjoint operator taken as a set of the resolvent. Thus, it is very interesting indeed to carry out a study on operators because at this juncture, going by the developments mentioned above, Rhaly [26] further remarked that if $A \in B(H)$ is to belong to the subclass of quasiposinormal, then A has to be normal and with the presence of $L \in B(H)$ as interrupter, $EE^* = E^*LE$, or equivalently, $[E^*, E] = E^*(I - L)E$. Here we see double observations which implies more requirements that L be hermitian and non-negative, EE^* hermitian, then every operator E in our sub-class should obey $E^*L^*E = E^*LE$; since $\langle LEf, Ef\rangle = \langle E^*LEf, f\rangle = \|E^*f\|^2\ \forall\ f$ and interrupter L is positive on $RanE$ (the range of E) which is called a posinormal operators. Further on posinormal, there exists another class called coposinormal operator E which exists under the condition that E^* is posinormal. Itoh [13] studied posinormality operators by their characterizations and spectral properties given by Rhaly[25] and gave an extension of the characterization of posinormal operators. Further, p-posinormal operators were introduced of which a study was carried on them, showing that the same operators (p-posinormal) are M-paranormal. Lee [11] studied posinormality and used the results to give a conjecture on indices of p-posinormal operators where it was shown that E is p-positive normal then E^n implies p-posinormality for all non-negative integers n was given. Further, a result was given that $E = U|E|$ is p-positive normal for $0 < p < 1$, the transformation of Aluthge $\widetilde{E} = |E|^{\frac{1}{2}}U|E|^{\frac{1}{2}}$ is $(p + \frac{1}{2})$-positive normal where T as an operator can be decomposed to $E = U|E|$ with U an isometry that is partial and $|E|$ is the square root of E^*E with $Y(U) = Y(|E|)$ and the condition that is kernel $Y(U) = Y(|E|)$ determining U uniquely and the decomposition of polar E was as well given. Duggal and Kubrusly [10] investigated Weyl's theorems for posinormal operators. The study was possible after presenting a survey on posinormal operators in which two classical problems restricted to this class of operators were considered. Since transitive operators are quasi-invertible and since invertible operators are posinormal, unique factorization for invertible transitive operators were given and a characterization for transitive totally hereditarily normaloid contractions with compact defect operator was proved. Moreover, the conditions for dominant operators to satisfy Weyl's theorem were given alongside exhibiting counterexamples to three incorrect statements of current literature on posinormality of operators was given. Sivamani [39] studied posinormality of which was extended to *para-normal, quasi-positive normal and quasi * para-normal operators on composition of space in Fock, were characterized. Much of this study has been done on the properties on the operators that are Hardy, Berg-man, and spaces in Bloch on the plane that is complex or ball with unit in C^n. Further, bounded and compact composition operator on the Fock were discussed alongside some classes of composition. The spaces in Fock F is a space in Hilbert of functions that are holomorphic on C^n with intrinsic product $\langle x, y\rangle = \frac{1}{(2\pi)^h} \int_{C^h} x(z)\overline{y(z)}e^{\frac{-1}{2}|y|^2}\ dl(z)$ where l represents a measure on C^n Lebesgue. Rhaly [24] did an introduction of a super class of the positive normal operators which were called supraposinormal operators for which the determination of conditions that sufficiently qualify supraposinormal operator to be posinormal and hyponormal, was carried out. Further, provided a short proof of a result that is well known, the hyponormality of C_k which refers to the generalization of order one Cesàro operators for $k \geq 1$ with an establishment of a connection that exists between this superclass and some recent publications on conditions that sufficiently qualify a lower triangular matrix, that is factorable, to be hyponormally linear operator which are bounded on ℓ^2. Rhaly [25] developed on posinormal operators and in relation to all other operators discussed prior, established that an operator A such that $A \in B(H)$, is supra-positive normal if positive operators L and M on H exists, then $ELE^* = E^*ME$, with one of L, M has range that is dense. Are at times called interrupters. Okelo[21] established the characteristics of supraposinormal operators in dense norm-attainable classes . Going by this studies, another class of cosupraposinormality will come out clearly where following the pattern created in getting coposinormality, it suffices that A is cosupraposinormal under one condition that A^* is supraposinormal. Bachir [1] studied the Fugled-Putnam theorem of operators and used the concepts to extend it to certain posinormal operators. In the study, asymmetric perspective of the Fuglede-Putnam was considered to give proofs on a given positive normal operators. Further, as a consequence of this results, the induction of the range on derivations that are general of these sets of operators were proved to be orthogonal to their kernel. In [8] the author contributed to the study of the theorem of Fuglede-Putnam in relation to (p, q)-quasipositive normal together with p, q-Coposinormal operator purely in generalizing posinormal operators and coposinormal operators to non-normal operators. Further, the author proved the theorem to supra class posinormal oprators called supra-posinormal operators and co-supra class posinormal operators called cosupra-posinormal operators. Kumar and Kiruthika [36] showed that operators of T_n group operators that are sequential in (p, k)-Quasiposinormal operators that are convergent to the topology that is normal to T within the class, then, the function spectra, Weyl spectra, Browder spectra and essentially surjective spectra, are T continuous while, taking T to be a (p, k)- quasipositive normal and $\overline{\lambda} \in \pi_{00}(T^*)$, it suffices $\widetilde{T}$ to be a pole of a set that resolves to T^*. Further, it was depicted that if there exists a continuous spectrum at $T \in B(H)$, then the spectrum is also continuous at T. Further, an analogy for a sequence T_n in (p, k)- quasiposinormal which is convergently normed to T, it suffices that the spectrum is also T continuous and T^*, which was referred to a point of continuity of σ_{ea}. Obogi, Asamba and Okelo [20] carried a research on posinormality operators with an emphasis on the characterization of numerical ranges. In the study, H was a space of complex Hilbert with inner product $\langle ., . \rangle$ equipment and $B(H)$ an algebraic operator that is linear and bounded acting on H. The range of numerals of linear operator A that is complex, linear and bounded on a space H of Hilbert, is a set $W(A) = \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}$. The radius of numerals $A \in B(H)$ was used to investigate the numerical range operators acting on spaces of Hilbert that are complex. Characterization of numerical ranges on positive normal operators for Hilbert spaces that are dimensionally infinite and are complex for a posinormal operator $A, W(A)$ to be nonempty, always and is an ellipse whose foci are the eigenvalues of A, were shown. Trieu and Rhaly [40] studied coposinormality in relation to the Cesàro matrices. The study showed that the matrices of Cesàro for any orders are co-posinormal by employing positive normality of interrupter that is diagonalized and uses the algorithm of Zeilberger assisted by maple computation. Veluchamy and Thulasimani [41] still worked on posinormality operators where the authors emphasized on factorization of posinormal operators in Hilbert spaces. Mecheri [15] studied Weyl's theorem where by they defined an operator to be Weyl if the index of Fredholm is zero. Hence, this concept was used to give the generalized theorem for posinormal operators. It was shown that the generalized theorem for $f(A)$ holds if A is conditionally totally positive normal or totally positive normal function $f_\circ$ that is analytical in a neighbourhood of $\sigma(A)$ that is open. Further, the author gave results of totally posinormal operator in relation to generalized Weyl's theorem whenever it holds or not. Rwenyo, Sabasi and Okelo [32] ventured their study in further characterization of posinormal operators whereby they established norm inequalities for posinormal operators and characterized further posinormal operators. The results obtained from this study were used to determine the areas in which posinormal operators are applicable in other mathematically related fields such as in representation of quantum observances and investigating the spectrum of various linear operators. Gopal [4] studied posinormality operators and used the knowledge to give an analogy on k-quasiposinormal weighted composition operators. For integer k that is non-negative, an operator A is k-quasiposinormal if $A^{*k}(AA^*)A^k \leq c^2 A^{*(k+1)}A^{(k+1)}$ for some $c > 0$. Further, the author described sufficiently the conditions that qualify operators that fall under weighted composition to be k-quasiposinormal operators. Sekar, Seshaiah, Senthil and Naik [37] studied quasi class operators where they studied Browder and a-Browder for k-quasi-*-class a operators. In the study, a k-quasi-*-class A operator E on the space of Hilbert H is complex if $E^{*k}(|E^2| - |E^*|^2)E^k \geq 0$ with k a number that is natural with the proofs of Browder, Browder's generalization, a-Browder's, and the theorem of a-Browder for k-quasi-*-class A operators were given. Janfada and Maleki [12] extended the theorem of Fuglede-Putnam where they showed that if E is k-quasihyponormal such that $EX = XF$, $\forall$ $E \in C_2(H)$, then $E^* = XF^*$. Also, it was clear that these class of operators induces the generalization of the kernel of inner derivations and the range that are orthogonal. For complex spaces of Hilbert H and G, $B(H, G)$ represents the space of linear operators that are bounded from H to G. In case when H and G were identified $B(H, G)$ was shortened to $B(H)$, not forgetting that $C_2(H)$ and C_1H denoted the class of Hilbert-Schmidt and the class of trace operators respectively to $B(H)$. Further, a two-sided -$*$-ideal is formed in $C_2(H)$ and $B(H)$ with $C_2(H)$ being a space of Hilbert with the inner product defined by $\langle E, F\rangle = \Sigma\langle E_{e_i}, F_{e_i}\rangle = tr\langle F^*E\rangle = tr\langle EF^*\rangle$ with $\{e_i\}$ a basis of H that is orthonormal and $tr(.)$ stands for a trace that is natural on $C_1(H)$. So a normed operator X with Hilbert-Schmidt in $C_2(H)$ is represented by $\|E\|_2 = \langle E, E\rangle^{\frac{1}{2}}$. Okelo [21] studied operators in general and used the concept to give analogy of norm-attainability of some elements. The author presented new findings on conditions that sufficiently that qualify norm-attainable to be Hilbert space operators. Moreover, conditions for norm-attainability for elementary operators and generalized derivations were also established. The main results showed that for $E \in B(H)$, $\psi \in w_0(E)$ and $\varphi > 0$, then an operator $G \in B(H)$ exists for $\|E\| = \|G\|$ for as long as $\|E - G\| < \varphi$. Furthermore, $\exists$ § $\in H$, $\|G§\| = \|G\|$ with $\langle Z§, §\rangle = \psi$. Nyakiti, Okongo and Okelo [19] studied on projective tensor norms and norm attainable α-derivation. They showed that if $\mu = \Sigma_i a_i \otimes b_i$ belongs to $V_\Gamma \otimes_p W_\Gamma$ and δ_N on μ is a norm-attainable α-derivation given by $\delta_N = \delta_N^{(1)} + \delta_N^{(2)}$ then, $\|\delta_N\| \leq \|\delta_N^{(1)} + \delta_N^{(2)}\| \leq 2\|\alpha_N\|$ holds. Hong, Wang and Gao [9] studied the norms of Hilbert spaces and used the concepts to give an analogy of norms of elementary operators. They gave a prove that the least upper bound of $\{\|\sum_{i=1}^n A_iXB_i\| : X \in B(H), \|X\| \leq 1\} = \sup\{\|\sum_{i=1}^n A_iEB_i\| : EE^* = E^*E = I, E \in B(H)\}$. Moreover, a proof that an operator X_0 exists with $\|X_0\| = 1$ such that $\|\sum_{i=1}^n A_iX_0B_i\| = \sup\{\|\sum_{i=1}^n A_iXB_i\| : X \in B(H), \|X\| \leq 1$ if and only if there exists a unitary $U_0 \in B(H)$ such that

$\|\sum_{i=1}^{n} A_i U_0 B_i\| = \sup\{\|\sum_{i=1}^{n} A_i X B_i\| : X \in B(H), \|X\| \leq 1\}$ Nathan [17] studied supercyclic operators and showed that operators that are linear on spaces of Hilbert whose subspaces have dimension n with an orbit that is dense and not of dimension $(n-1)$, leads to the discovery of operators referred to the n-super-cyclic. Also results for operators that are co-hyponormal being n-super-cyclic were given. Further, it was proved there exists n circles at the central point of the origin in which there is an intersection to one of these circles by each component of the spectra for an n-super-cyclic operator. Sid [38] studied quasinormal operators and gave an extension of the concept to properties that are normal, obeys the property of Bishop on the spaces of Hilbert, as well proved the characteristics of n-power quasi-normal the operators, $T \in [nQN]$, described in the publication of Sid Ahmed of the year 2011. Particularly, it was shown that the property of an invariant translation is satisfied by operator $T \in [nQN]$ which when not invertible, it isn't super-cyclic. Also, a sub-scalar $T \in [2QN]$ ordered m was studied to be equivalent to restriction of operators in specific, scalar operators of order M to a sub-space that is invariant. Senthilkumar [34] and Kiruthika [35] studied the continuity property of (p,k)-quasipositive normal and (p,k)-quasi-hyponormal operators which are not the spectrum of posinormal operators. Some properties such as characteristics of posinormal operators and supraposinormal operators together with their relationships in norm attainable classes have not been fully investigated. Therefore, the study of supraposinormal operators and posinormal operators is very crucial more so in terms of their characteristics that relates one operator to another. Particularly, the discipline is connected and applied to quantum theory among others. All these constitute a non-passive study in supraposinormal operators. This study therefore is to characterize the supraposinormal operators and in norm-attainable classes.

2 Preliminaries

This sections provides some definitions which are useful in the sequel.

Definition 2.1 ([4], Definition 3.10) *An operator $E \in B(H)$ is said to be normal if $EE^* = E^*E$, where, E^* is the adjoint of E.*

Definition 2.2 ([9], Definition 3.6) *An operator E is linear if for f, g and scalar ψ, $E(f+g) = E(f) + E(g)$ and $E(\psi f) = \psi E(f)$.*

Definition 2.3 ([3], Definition 2.2) *If $E \in B(H)$, then E is positive normal if for $X > 0$, $EE^* = E^*XE$. An operator E is called co-posinormal if and only if E^* is positive normal.*

Definition 2.4 ([24], Definition 2.5.) *If $E \in B(H)$, then E is supraposinormal if for $X, M > 0$, $EXE^* = E^*ME$. An operator E is called co-supraposinormal if and only if E^* is supraposinormal.*

Definition 2.5 ([33], Definition 1.1) *An operator $E \in B(H)$ is quasi-posinormal operator if $E^*(E^*E)E \leq E^*(\gamma^2(EE^*))E$ or simply if the $Range(E^2) \subseteq Range(E^*)$.*

Definition 2.6 ([3], Definition 3.10) *An operator E is hermitian if $E = E^*$.*

Definition 2.7 ([25], Definition 2.1.) *An operator E is hyponormal if $\|E^*y\| \leq \|Ey\|$ for every $y \in H$.*

Definition 2.8 ([26], Definition 2.4.) *An operator E is cohyponormal if E^* is hypornormal.*

Definition 2.9 ([30], Definition 2.7.) *An operator $E \in B(H)$ is called dominant if $(T - \lambda)(T - \lambda)^* = M_\lambda(T - \lambda)^*(T - \lambda)$, for each $\lambda \in \mathcal{M}$ and $\mathcal{M}_\lambda > 0$.*

Definition 2.10 ([26], Definition 2.3.) *An operator E is said to be semi - hyponormal if it is both hyponormal and cohyponormal.*

Definition 2.11 ([17], Definition 3.2) *Two operators $e, g \in H$ are orthogonal if $\langle e, g\rangle = 0$, denoted by $e \perp g$.*

3 Main results

In this section, we give the main results of this study. First, we state some auxiliary results before embarking on the main results.

Proposition 3.1 *Let S be a supraposinormal operator on a separable Hilbert space H with $SPS^* = S^*QS$ and $P \geq 1 \geq Q \geq 0$. Then S is posinormal.*

Proof. Let S be supraosinormal such that $PS^* = QS$. Squaring S^* on the left side and S on the right side of we have $P\{S^*\}^2 = Q\{S\}^2$ such that when expanded we will have $PS^*S^* = QSS$. Adjoining S^* on the side of left and S on the side of right we will have $PSS^* = QS^*S$. Rearranging the equation we will have $SPS^* = S^*QS$. If we let P to be invertible such that $PP^{-1} = I$, the equation will be $SS^* = S^*QS$ which is a posinormal operator. Hence, S is posinormal.

Proposition 3.2 *Let S be a supraposinormal operator on a separable Hilbert space H with $SPS^* = S^*QS$ and $P \geq 1 \geq Q \geq 0$. Then S is coposinormal.*

Proof. For coposinormality, we show that S^* is posinormal. Let, $PS^* = QS$. Squaring both sides we have $(PS^*)^2 = (QS)^2$ which expands to $(PS^*)(PS^*) = (QS)(QS)$. Factorizing P on the left and Q on the right we have $P(S^*S^*) = Q(SS)$. Adjoining S^* on the left and S on the right we have $P(S^*S) = Q(SS^*)$. Rearranging the equation we have $S^*PS = SQS^*$. Taking P to be invertible we will have $S^*S = SQS^*$. This shows that S^* is posinormal, hence proving coposinormality.

Proposition 3.3 *Let S be a supraposinormal operator on a separable Hilbert space H with $SPS^* = S^*QS$ and $P \geq 1 \geq Q \geq 0$. Then S is hyponormal.*

Proof. We let S be supraposinormal such that $S^*PS = SQS^*$ and we know that $S^*PS = SQS^*$ is posinormal for as long as P is invertible such that it becomes $S^*S = SQS^*$ which can also be given as $S^*S = \lambda^2 SS^*$, for as long $Q = \lambda^2$. If $M = \lambda^2$, the same equation becomes $S^*S = MSS^*$, for all $M > 0$. The equation qualifies to be M-hyponormal of which for $x \in H$ and $S \in B(H)$, the same equation can be given by $\| Sx \| \leq M \| S^*x \|$. Squaring both sides we have

$$(\| Sx \|)^2 \leq (M \| S^*x \|)^2.$$

When we expand both sides the equation becomes

$$Sx, Sx \leq M^2 S^*x, S^*x.$$

Adjoining Sx on the left and S^*x on the right, the equation becomes

$$S^*Sx, x \leq M^2 SS^*x, x.$$

Rearranging gives

$$S^*Sx, x - M^2 SS^*x, x \leq 0.$$

Since $M > 0$, we take an arbitrary value such that $M = 1$ which makes the equation to become

$$S^*Sx, x - 1^2 SS^*x, x \leq 0$$

which brings the equation to be

$$S^*Sx, x - SS^*x, x \leq 0.$$

On factorizing the equation we will have

$$\langle \{S^*S - SS^*\}x, x\rangle \leq 0.$$

Proposition 3.4 *Let S be a supraposinormal operator on a separable Hilbert space H with $SPS^* = S^*QS$ and $P \geq 1 \geq Q \geq 0$. Then S is dominant.*

Proof. Let S be supraposinormal such that $SPS^* = S^*QS$. Now, S being supraposinormal is posinormal whenever the operator P is invertible if $PP^{-1} = P^{-1}P = I$, such that the equation becomes $SS^* \leq \lambda^2 S^*S$ for $Q = \lambda$ for which $\lambda \geq 0$. Further, we have shown that it is hyponormal for which $\lambda = 1$ such that the equation becomes $SS^* \leq S^*S$. We use the equation, $\| (S - \lambda)^*x \| \leq M_\lambda \| (S-\lambda)x \|$. Squaring both sides the equation becomes $[S - \lambda]^*x, [S-\lambda]^*x \leq M_\lambda[S-\lambda]x, M_\lambda[S-\lambda]x$. Adjoining $\{[S-\lambda]^*x\}$ on the left and $\{[S-\lambda]x\}$ on the right, we have the following equation $\{[S-\lambda][S-\lambda]^*x, x\} \leq \{M_\lambda^2[S-\lambda]^*[S-\lambda]x, x\}$. By the definition of C^*-algebra, we get that $\| a^*a \| = \| a \|^2$ applying this concept in our equation $\{[S-\lambda]^*x\}^2 \leq M_\lambda^2\{[S-\lambda]x\}^2$ since $\{[S-\lambda]x\}$ is an adjoint operator. Now we take square roots on both sides such that $\{[S-\lambda]^*x\}^{2\bullet\frac{1}{2}} \leq M_\lambda^{2\bullet\frac{1}{2}}\{[S-\lambda]x\}^{2\bullet\frac{1}{2}}$ which is simplified to $\{[S-\lambda]^*x\} \leq M_\lambda\{[S-\lambda]x\}$. Applying the Cauchy-Schwarz inequality becomes $\| \{[S-\lambda]^*x\} \| \leq \| \{M_\lambda[S-\lambda]x\} \|$.

Lemma 3.5 *Let $E \in NA(H)$ be a posinormal operator. If E is invertible, then E^{-1} is supraposinormal.*

Proof. Let E be posinormal such that for $p > 0$, then $P^2E^*E - EE^* \geq 0$, and adding EE^* both sides such that $P^2E^*E - EE^* + EE^* \geq 0 + EE^*$ makes the equation to be $P^2E^*E \geq EE^*$. Multiplying both sides by inverses of EE^* we have $E^{-1}(P^2)E^*E)(E^*)^{-1} \geq E^{-1}EE^*(E^*)^{-1}$. Simplifying the side of the right we have $E^{-1}(P^2E^*E)(E^*)^{-1} \geq (E^{-1}E)(E^*(E^*)^{-1})$ such that $E^{-1}(P^2E^*E)(E^*)^{-1} \geq 1$. Inversing both side as $[E^{-1}(P^2E^*E)(E^*)^{-1}]^{-1} \geq 1^{-1}$ which is $\frac{1}{E^{-1}(P^2E^*E)(E^*)^{-1}} \geq 1$, further becoming to $E(\frac{1}{P^2}(E^*)^{-1}E^{-1})E^* \geq 1$. This makes the side of the left to be less or equal to that of the right such that $E(\frac{1}{P^2}(E^*)^{-1}E^{-1})E^* \leq 1$. Dividing both sides by EE^* we will have $(\frac{1}{P^2}(E^*)^{-1}E^{-1}) \leq \frac{1}{EE^*}$ which may be given as $(\frac{1}{P^2}(E^*)^{-1}E^{-1}) \leq E^{-1}(E^*)^{-1}$. If we find the reciprocal of $\frac{1}{P^2}$ while we commute other operators we have $(P^2E^{-1}(E^*)^{-1}) \leq (E^*)^{-1}E^{-1}$ making the left larger or equal to the right such that, $(P^2E^{-1}(E^*)^{-1}) \geq (E^*)^{-1}E^{-1}$. Finding the square root of P^2 yields $(PE^{-1}(E^*)^{-1}) \geq (E^*)^{-1}E^{-1}$ of which if we introduce an operator $Q > 0$ on the side of the right we will have an equalized equation, thus becoming, $(PE^{-1}(E^*)^{-1}) = Q(E^*)^{-1}E^{-1}$. Rearranging P and Q becomes $(E^{-1}P(E^*)^{-1}) = (E^*)^{-1}QE^{-1}$. Implying that E^{-1} is supraposinormal.

Theorem 3.6 *Let $E \in NA(H)$ be a supraposinormal operator. If E invertible, then E^{-1} is posinormal.*

Proof. Let E be an invertible supraposinormal for which $V, U > 0$ and $x_n \in NA(H)$ $V(E^*x_n) = U(Ex_n)$. Taking inverses both sides makes the equation to be $[V(E^*x_n)]^{-1} = [U(Ex_n)]^{-1}$. Squaring both sides, that is,

$$([V(E^*x_n)]^{-1})^2 = ([U(Ex_n)]^{-1})^2$$

in which when we expand the equation, we will have

$$[V^{-1}(E^*x_n)^{-1}, V^{-1}(E^*x_n)^{-1}] = [U^{-1}(Ex_n)^{-1}, U^{-1}(Ex_n)^{-1}]$$

which can be written as

$$V^{-1} \cdot V^{-1}[(E^*x_n)^{-1}, (E^*x_n)^{-1}] = U^{-1} \cdot U^{-1}[(Ex_n)^{-1}, (Ex_n)^{-1}].$$

But by indices $V^{-1} \cdot V^{-1} = V^{-2}$ and $U^{-1} \cdot U^{-1} = U^{-2}$ which substituted in the equation we will have

$$V^{-2}[(E^*x_n)^{-1}, (E^*x_n)^{-1}] = U^{-2}[(Ex_n)^{-1}, (Ex_n)^{-1}]$$

which can be written as

$$\frac{1}{V^2}[(E^*x_n)^{-1}, (E^*x_n)^{-1}] = \frac{1}{U^2}[(Tx_n)^{-1}, (Ex_n)^{-1}].$$

Taking the reciprocal of U^{-2} and V^{-2} and commuting the other operators,we will have $V^2[(E^*x_n)^{-1}, (E^*x_n)^{-1}] = U^2[(Ex_n)^{-1}, (Ex_n)^{-1}]$. Adjoining $(E^*x_n)^{-1}$ on the left and $(Ex_n)^{-1}$ on the right we will have

$$V^2[(E^*x_n)^{-1}, (Ex_n)^{-1}] = U^2[(Ex_n)^{-1}, (E^*x_n)^{-1}].$$

Letting U to be invertible such that $UU^{-1} = I$, will have

$$V^2[(E^*x_n)^{-1}, (Ex_n)^{-1}] = [(Ex_n)^{-1}, (E^*x_n)^{-1}]$$

which makes the left larger or equivalent right, such that

$$V^2[(E^*x_n)^{-1}, (Ex_n)^{-1}] \geq [(Ex_n)^{-1}, (E^*x_n)^{-1}],$$

showing that E^{-1} is posinormal.

Theorem 3.7 *Let $E \in NA(H)$ be supraposinormal. If E is invertible with invertible interrupters (A, B), then its inverse E^{-1} is posinormal.*

Proof. Let E be supraposinormal such that $EAE^* = E^*BE$. Since it is invertible, then there exists inverses of E and E^* which when applied on both sides we have

$$(E^*)^{-1}EAE^*E^{-1} = (E^*)^{-1}E^*BEE^{-1}$$

which can be written as $(E^*)^{-1}EAE^*E^{-1} = [(E^*)^{-1}E^*]B[ET^{-1}]$. But $EE^{-1} = I$ and $E^*E^{*-1} = I$ which when substituted on the side of right, the equation becomes $(E^*)^{-1}EAE^*E^{-1} = IBI$, implying $(E^*)^{-1}EAE^*E^{-1} = B$. We take inverses on either side, we will have $E^*E^{-1}A^{-1}(E^*)^{-1}E = B^{-1}$ while we divide both sides by E^* and E we have $E^{-1}A^{-1}(E^*)^{-1} = (E^*)^{-1}B^{-1}E^{-1}$. Letting A^{-1} be invertible, we have $E^{-1}(E^*)^{-1} = (E^*)^{-1}B^{-1}E^{-1}B$ is an interrupter and hence it is positive, therefore its inverse B^{-1} is also positive. Hence, E^{-1} is posinormal.

Corollary 3.8 *Let $E \in NA(H)$ be a dominant supraposinormal operator, then it is posinormal.*

Proof. Let E be a dominant supraposinormal operator such that for $\lambda, \mu \in \mathcal{C}$, $Q_\mu \parallel (E-\lambda)^* x_n \parallel = P_\mu \parallel (E-\lambda)x_n \parallel$ for $x_n \in NA(H)$. Squaring both sides we have

$$(Q_\mu \parallel (E-\lambda)^* x_n \parallel)^2 = (P_\mu \parallel (E-\lambda)x_n \parallel)^2$$

which becomes

$$(Q_\mu(E-\lambda)^* x_n, Q_\mu(E-\lambda)^* x_n = (P_\mu(E-\lambda)x_n, P_\mu(E-\lambda)x_n)$$

and which can further be expressed as

$$(Q_\mu \cdot Q_\mu)(E-\lambda)^* x_n, (E-\lambda)^* x_n = (P_\mu \cdot P_\mu)(E-\lambda)x_n, (E-\lambda)x_n).$$

But $Q_\mu \cdot Q_\mu = (Q_\mu)^2$ and $P_\mu \cdot P_\mu = (P_\mu)^2$ which when substituted we have

$$Q_\mu^2(E-\lambda)^* x_n, (E-\lambda)^* x_n = P_\mu^2(E-\lambda)x_n, (E-\lambda)x_n).$$

Adjointing $E-\lambda)^* x_n$ on the side of left and $E-\lambda)x_n$ on right we have

$$Q_\mu^2(E-\lambda)^* x_n, (E-\lambda)x_n = P_\mu^2(E-\lambda)x_n, (E-\lambda)^* x_n).$$

Taking the square roots of Q_μ^2 and P_μ^2 we have

$$Q_\mu(E-\lambda)^* x_n, (E-\lambda)x_n = P_\mu(E-\lambda)x_n, (E-\lambda)^* x_n).$$

Rearranging Q_μ and P_μ we have

$$(E-\lambda)^* x_n Q_\mu(E-\lambda)x_n = (E-\lambda)x_n P_\mu(E-\lambda)^* x_n).$$

Taking Q_μ to be invertible such that $Q_\mu Q_\mu^{-1} = 1$, then

$$(E-\lambda)^* x_n, (E-\lambda)x_n = (E-\lambda)x_n P_\mu(E-\lambda)^* x_n)$$

which is a posinormal equation.

Corollary 3.9 *Let $E \in NA(H)$ be p-supraposinormal, then E is posinormal.*

Proof. Let E be p-supraposinormal such that for $S, M > 0$ we have $(SE^*)^P = (ME)^P$. Squaring both sides we get $[(SE^*)^2]^P = [(ME)^2]^P$ which when expanded we have $(SE^*, SE^*)^P = (ME, ME)^P$ which in another form is expressible as $[S \cdot S(E^*, E^*)]^P = [M \cdot M(E, E)]^P$. But $[S \cdot S] = S^2$ and $[M \cdot M] = M^2$ which when substituted we have $[S^2(E^*, E^*)]^P = [M^2(E, E)]^P$. Adjoining E^* on the side of the left and E on the side of the right we have $[S^2(E, E^*)]^P = [M^2(E^*, E)]^P$. Taking $P = I$ we have $[S^2(E, E^*)]^I = [M^2(E^*, E)]^I$ which implies $S^2(E, E^*) = M^2(E^*, E)$. Taking square root of S^2 and M^2 we make the equation to be $S(E, E^*) = M(E^*, E)$ alongside rearranging S and M we have $(ESE^*) = (E^*ME)$. Taking S to be invertible such that $SS^{-1} = I$ we shall have the same equation as $(E, E^*) = (E^*ME)$ which is a posinormal operator.

4 Open Problems

The results obtained in this work are specific to the supraposinormal operators on complex Hilbert spaces. Can characterization of cosupraposinormal operators when H is complex and norm-attainable suffice? Moreover, can these results be extended in a general complex Banach space setting?

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