Int. J. Open Problems Compt. Math., Vol. 15, No. 1, March 2022
Print ISSN: 1998-6262, Online ISSN: 2079-0376

Maximal Chain Properties for Mappings in Bitopological Spaces

Benard Okelo

Department of Pure and Applied Mathematics,
Jaramogi Oginga Odinga University of Science and Technology,
Box 210-40601, Bondo-Kenya.
e-mail: benard@aims.ac.za

Received 12 April 2021; Accepted 2 February 2022

Abstract

The fundamental mathematical problem that remains unsolved is the generalization of maximality chain conditions for chains of homomorphisms in topological spaces. We give maximal chain properties in bitopological spaces with consideration to elements which are unique and non-isolated.

Keywords: *Maximality, Bitopological space, Chain, Uniqueness*
2010 Mathematics Subject Classification: 49J27, 49J45, 54H25, 54E15.

1 Introduction

Considerations have been given to homomorphisms on topological spaces and their properties have been characterized for along period of time with interesting results obtained(see [4] - [14] and the references therein). There exits a fundamental mathematical problem that exists on maximality conditions for chains in general topological spaces [1], [2], [3] and [15] which is: *Does there exist unique non-isolated maximal elements in a topological space that satisfy strong convergence in a general setting?.* Many considerations have been given in special cases as solving this problem in a general setup has been elusive since the underlying structures in various space are intricate and varies from one space to another. This is the question that this paper addresses. Below we frame is the main theorem of this work.

Theorem 1.1 *Every convergent sequence in a bitopological space B is Cauchy and converges to a unique maximal non-isolated element in B.*

Before we move to the proof of our main result, we first provide some preliminary notes which are useful in the sequel in the next section.

2 Preliminaries

This sections provides some definitions and notations. Throughout this work we denote lower semi-continuous and upper semi-continuous by *lsc* and *usc* respectively.

Definition 2.1 *Let B be a non-empty set and τ_1, τ_1 be topologies on B. Then the ordered triple (B, τ_1, τ_1) is called a bitopological space.*

Remark 2.2 *For simplicity we denote a bitopological space (B, τ_1, τ_1) by B unless otherwise stated.*

Definition 2.3 *Consider the homomorphism $J\colon B \times B \to [0, \infty)$ on a bitopological space B. Then $\{\xi_n\}_{n \in \mathcal{N}}$ is a Cauchy sequence in a metrizeable bitopological space B, if for every $\epsilon > 0$ there exists an $n_0 \in \mathcal{N}$ such that each $m, n \in \mathcal{N}$, $J(\xi_n, \xi_m) < \epsilon$, whenever $n_0 < m < n$.*

We extend the notion of unique and non-isolated element for generalized pseudo-metric spaces found in [2] to the case of bitopological spaces.

Definition 2.4 *Consider the bitopological space B an a mapping $J\colon B \times B \to [0, \infty)$. We call J unique and non-isolated in B if the following conditions are satisfied:*

(i). $J(z, x) \leq J(z, y) + J(y, x), \quad for\ \ all, x, y, z \in B,$

(ii). $J(\cdot, x)$ *is lsc(respectively usc),* $\quad x \in B,$

(iii). Each Cauchy sequence in B is a Cauchy subsequence in B_n.

The main results of this work follows immediately in the next section.

3 Main results

Now, we give the main results of this study. First, we state some auxiliary results and we express our conditions by removing the compactness condition on B as follows.

Proposition 3.1 *Let B be a bitopological space. Then every sequence in B is convergent.*

Proof. The proof is trivial and equally follows the proof of convergence conditions in [1].

Proposition 3.2 *Let B be a uniform bitopological space. Then any B-unique and non-isolated J on B satisfies $J(a,b) = 0$ and $J(b,a) = 0$ which yields $b = a$, for all $a, b \in B$.*

Proof. Suppose that $J(a,b) = 0$ and also $j(b,a) = 0$. Fix $a_{2r-1} = a$ and $a_{2r} = b$, for some positive $r \in \mathcal{N}$. It is easy to see that, $J(\xi_n, \xi_m) < \epsilon$ holds for each $m \neq n$ whereby $m, n \in \mathcal{N}$, and $\{\xi_n\}_{n\in\mathcal{N}}$ is Cauchy in B by Condition (iii) in Definition 2.4, i.e. $a = b$. This completes the proof.

Remark 3.3 *Suppose that $J: B \times B \to \mathcal{R}$ and $K: B \to \mathcal{R}$ are homomorphisms and J satisfies Condition (i) in Definition 2.4, then the following relation defines transitivity condition $\propto$:*

$$b \propto a \text{ iff } \psi(b) + J(b,a) - K(a) \leq 0, \quad for \ \ alla, b \in B. \tag{1}$$

The following lemma is an analogy of uniqueness for Bergman spaces which are not necessarily uniform.

Lemma 3.4 *Suppose that B is a complete bitopological space and J be B-unique and non-isolated in B. Suppose also that $K: B \to \mathcal{R}$ be a lsc homomorphism bounded below. Let $V \subset B$ be a nonvoid maximal chain (for $\propto$) such that $J(a,b) = 0$ or $J(b,a) = 0$, for each $a, b \in V$, $a \neq b$. Then V has a unique non-isolated maximal element.*

Proof. Maximality of the element follows from [15]. Uniqueness and the element being non-isolated follows from [12]. Now, by convergence conditions of Proposition 3.1, the zero property of 3.2 and Transitivity of Remark 3.3, the proof is complete.

Lemma 3.5 *For every complete bitopological space B, a map J on B tends to a maximal point in B.*

Proof. Since the sequences are Cauchy then completeness is guaranteed. The rest follows from 3.4. This completes the proof.
At this juncture, we proceed to proving Theorem 3.6.

Proof. We continue with **Proof of Theorem** 3.6 as follows. Since convergence is a requirement, this is guaranteed from Lemma 3.5. Other conditions follow immediately from Proposition 3.2 and Lemma 3.4. We only need to prove that every convergent sequence is Cauchy which is immediate from [2]. This completes the proof.

Corollary 3.6 *Every convergent sequence in a compact a bitopological space B is Cauchy and converges to a unique maximal non-isolated element in B.*

Proof. The proof follows analogously from the proof of Theorem 3.6 with the inclusion of compactness conditions for bitopological spaces.

4 Open Problem

Does Theorem 3.6 hold for N-topological spaces in general?

References

[1] P. R. Halmos, *A Hilbert space problem book*, Van Nostrand, New York, 1967.

[2] J. L. Kelley, *General Topology*, Springer Verlag, New York, 1975.

[3] B. Ogola, N. B. Okelo, and O. Ongati Certain Notions of Continuity in Bitopological Spaces, Int. J. Open Problems Compt. Math., **14**(4), (2021), 48–67.

[4] N. B. Okelo, *On Convex Optimization in Hilbert Spaces,* Maltepe J. Math., **2**(2019), 89–95.

[5] N. B. Okelo, J. O. Agure and D. O. Ambogo, *Norms of elementary operators and characterization of norm-attainable operators*, Int. J. Math. Anal., **24** (2010), 1197–1204.

[6] N. B. Okelo, *The norm-attainability of some elementary operators*, Appl. Math. E-Notes, **13** (2013), 1–7.

[7] N. B. Okelo, *α-Supraposinormality of operators in dense norm-attainable classes*, Universal Journal of Mathematics and Applications, **2** (2019), 42–43.

[8] N. B. Okelo, *On orthogonality of elementary operators in norm-attainable classes*, Taiwanese Journal of Mathematics, **24** (2020), 119–130.

[9] N. B. Okelo, M. O. Okongo and S. A. Nyakiti, *On projective tensor norm and norm-attainable α-derivations*, Int. J. Contemp. Math. Sciences, **5** (2010), 1969–1975.

[10] N. B. Okelo, J. O. Agure and P. O. Oleche, *Various notions of orthogonality in normed spaces*, Acta Mathematica Scientia, **33** (5) (2013), 1387–1397.

[11] N. B. Okelo, *On Norm-attainable operators in Banach Spaces,* J. Function Spaces, **1**(2020), 1–6.

[12] N. B. Okelo, *On certain conditions for convex optimization in Hilbert Spaces,* Khayyam J. Math., **5**(2019), 108–112.

[13] N. B. Okelo, Fixed points approximation for nonexpansive operators in Hilbert spaces, Int. J. Open Problems Compt. Math., **14**(1), (2021), 1–5.

[14] N. B. Okelo and J. Oburu, On Monopole Bundle Systems of Complex Hypermanifolds for Composition Operators, (2022), 1–10. https://arxiv.org/abs/2202.04440

[15] S. Suantai, Y. Shehu and P. Cholamjiak, *Nonlinear iterative methods for solving the split common null point problem in Banach spaces,* Optimization Methods and Software, **34**(2019), 853–874.