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Surface construction with a common involute line of curvature

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Abstract

In the present paper, we parametrically represent surfaces passing through an involute of a prescribed curve as a line of curvature. We express these surfaces by using the Frenet vector fields of the involute curve. We give the sufficient constraints for the coefficients of the Frenet vector fields. Finally, we present some illustrative examples.

Keywords: *Parametric Surface, Line of Curvature, Involute Curve.*
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1 Introduction

Curves and surfaces appear in every differential geometry book [1, 2, 3, 4]. Most existing work related with surfaces concentrated on forward analysis: given a surface, find and classify special curves on the surface in question. However, the most relevant problem is the reverse one: given a curve, construct surfaces possessing it as a common special curve. Wang et al. [5] handled the problem of construction of a surface family upon a given curve and obtained constraints for this curve to be a geodesic curve on every representative of the pencil. In 2011, authors [6] changed the given geodesic curve to a line of curvature and represented surfaces parametrically with a common line of curvature. Bayram et al. [7] obtained the necessary and sufficient constraint for a given curve to be an asymptotic curve on every member of a surface pencil. In [8], authors obtained similar results in the Galilean 3-space. Atalay and Kasap [9] studied surfaces with a common null asymptotic curve. Ergün et al. [10] constructed similar surfaces in 3-dimensional Minkowski space. Bayram and Bilici [11] expressed a surface pencil interpolating an involute curve as a common asymptotic curve.

This kind of studies find operations in computer graphics and image processing, variant industrial operations, such as cloth manufacturing, painting and cutting way, textile manufacturing, hose manufacturing, plain-metal-based manufacturing, architectural computer aided design, astrophysics and astronomy [12, 13, 14, 15, 16].

In this paper, we construct surfaces using an involute of a given curve and obtain sufficient condition such that it is a line of curvature on every member of this family. Finally, we present some illustrative examples.

2 Preliminaries

A curve $\alpha(s)$, $L_1 \le s \le L_2$, is called a parameter curve on a surface $P(s,t)$ in $\mathbb{R}^3$ if $P(s,t_0) = \alpha(s)$ for a fixed t_0. In the present study, we denote the derivative of α by α' and assume that α is a regular curve with nonvanishing acceleration, that is, $\alpha'(s) \neq 0 \neq \alpha''(s)$, $L_1 \le s \le L_2$. Then, the unit tangent, principal normal, and binormal vector fields of the curve at the point $\alpha(s)$ are defined by $T(s) = \alpha'(s)$, $N(s) = \frac{\alpha''}{\|\alpha''\|}$ and $B(s) = T(s) \times N(s)$, respectively. The set $\{T(s), N(s), B(s)\}$ is called the Frenet frame field along $\alpha(s)$. The following formula exist for the Frenet frame

$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix},$$

where $\tau(s) = -\langle B'(s), N(s)\rangle$ and $\kappa(s) = \|\alpha''(s)\|$ are the so called torsion and curvature of the curve $\alpha(s)$, respectively [2].

Let $\alpha(s)$ and $\beta(s)$, $L_1 \le s \le L_2$, be two curves. If β intersects the tangent vector fields of α orthogonally, then β is called an *involute* of α. Alternatively, the orthogonal trajectory of the tangent vector fields of the curve α is called an *involute* of the curve α. An involute of a curve $\alpha(s)$ with unit speed is given by

$$\beta(s) = \alpha(s) + (c - s)\,T(s), \tag{1}$$

where c is a constant real number and $T(s)$ is the unit tangent vector field of the curve $\alpha(s)$ [4]. For consistency, we assume that $c - s \neq 0,\ L_1 \leq s \leq L_2$.

When a rigid body moving along an arclength curve $\alpha(s)$, the motion in question contains translation along and rotation about α. The rotation is characterized by an angular velocity vector D that satisfies $T' = D \times T$, $B' = D \times B$ and $N' = D \times N$. The vector D is called the *Darboux vector*. Darboux vector is given by $D = \tau T + \kappa B$, in terms of Frenet vector fields $T,\ N$ and B [3]. Also, we have $\kappa = \|D\| \cos\theta,\ \tau = \|D\| \sin\theta$, where θ is the angle between the binormal vector field $B(s)$ and the Darboux vector of α and $0 \leq \theta < \frac{\pi}{2}$. Observe that $\theta = \arctan \frac{\tau}{\kappa}$.

Let $\alpha(s),\ L_1 \leq s \leq L_2$, be an arclength curve and $\beta(s),\ L_1 \leq s \leq L_2$, be an involute of α. Then one has

$$\begin{pmatrix} T^*(s) \\ N^*(s) \\ B^*(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{2}$$

where $\{T^*(s), N^*(s), B^*(s)\}$ and $\{T(s), N(s), B(s)\}$ are Frenet frame fields of the curves β and α, respectively, and θ is the angle between the binormal vector field $B(s)$ of α and the Darboux vector D.

3 Main results

Assume that we have a unit speed parametric space curve $\alpha(s),\ L_1 \leq s \leq L_2$, and $\|\alpha''(s)\| \neq 0,\ L_1 \leq s \leq L_2$. Let $\beta(s),\ L_1 \leq s \leq L_2$, be an involute of the given curve $\alpha(s)$.

Surfaces interpolating $\beta(s)$ are given in the parametric form as

$$P(s,t) = \beta(s) + u(s,t)\,T^*(s) + w(s,t)\,B^*(s) + v(s,t)\,N^*(s), \tag{3}$$

where $u(s,t),\ w(s,t)$ and $v(s,t)$ are real valued C^1 functions. They are called *marching-scale functions* and $\{T^*(s), N^*(s), B^*(s)\}$ is the Frenet frame field of the curve β. With the help of Eqn. (2) we may express Eqn. (3) in terms of Frenet frame field $\{T(s), N(s), B(s)\}$ of the curve α as

$$\begin{aligned} P(s,t) &= \beta(s) + (w(s,t)\sin\theta - v(s,t)\cos\theta)\,T(s) \\ &\quad + (v(s,t)\sin\theta + w(s,t)\cos\theta)\,B(s) + u(s,t)\,N(s), \end{aligned} \tag{4}$$

where $L_1 \leq s \leq L_2,\ T_1 \leq t \leq T_2$.

Remark 3.1 *Note that changing marching-scale functions results in different surfaces interpolating $\beta(s)$ as a common curve.*

Our aim is to find the sufficient constraints for which the curve $\beta(s)$ is a parameter curve and a line of curvature on the surface $P(s,t)$. First of all, since $\beta(s)$ is a parameter curve on the surface $P(s,t)$, we have

$$u(s,t_0) = w(s,t_0) = v(s,t_0) \equiv 0,\ L_1 \le s \le L_2,\ \exists t_0 \in [T_1, T_2]. \tag{5}$$

Secondly, we will find the constraint such that the curve β is a line of curvature on the surface $P(s,t)$. The normal vector field of $P(s,t)$ can be expressed as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

Using Eqns. (2) and (4), the normal vector field can be expressed along the curve β as

$$n(s,t_0) = \phi_1(s,t_0)\,T^*(s) + \phi_2(s,t_0)\,B^*(s) + \phi_3(s,t_0)\,N^*(s), \tag{6}$$

where

$$\begin{cases} \phi_1(s,t_0) \equiv 0,\ \phi_2(s,t_0) = \kappa(c-s)\frac{\partial v}{\partial t}(s,t_0), \\ \phi_3(s,t_0) = -\kappa(c-s)\frac{\partial w}{\partial t}(s,t_0), \end{cases}$$

κ is the curvature of the curve α.

Theorem 3.2 *If the surface normals through a surface curve cast a developable ruled surface, then it is a line of curvature on that surface* [2].

Let $\psi(s,t) = \beta(s) + tn_1(s)$ be a surface where $n_1(s) = \sin\varphi B^* + \cos\varphi N^*$, the vector functions $B^*,\ N^*$ are the binormal and the pirincipal vector fields of $\beta(s)$, respectively. According to Theorem 3.2, $\beta(s)$ is a line of curvature on the surface $P(s,t)$ if and only if $\psi(s,t)$ is developable and $n(s,t_0)$ is paralell to $n_1(s)$. The surface $\psi(s,t)$ is developable if and only if

$$\det(\beta', n_1, n_1') = 0 \Leftrightarrow \theta(s) + \varphi(s) = \text{constant}.$$

$n(s,t_0)$ is paralell to $n_1(s)$ if and only if

$$\phi_2(s,t_0) = \lambda(s)\sin\varphi,\ \phi_3(s,t_0) = \lambda(s)\cos\varphi,$$

where $L_1 \le s \le L_2,\ T_1 \le t, t_0 \le T_2$ (t_0 fixed), $\lambda(s) \ne 0,\ \varphi$ is the angle between the principal normal vector field $N^*(s)$ of β and the surface normal vector field $n(s,t_0)$.

Theorem 3.3 *Let $\alpha(s),\ L_1 \le s \le L_2$, be an arclength curve with nonzero curvature and $\beta(s),\ L_1 \le s \le L_2$, be its involute. β is a line of curvature on the surface* (3) *if*

$$\begin{cases} u(s,t_0) = w(s,t_0) = v(s,t_0) \equiv 0, \\ \theta + \varphi = \textit{constant},\ \phi_2(s,t_0) = \lambda(s)\sin\varphi,\ \phi_3(s,t_0) = \lambda(s)\cos\varphi, \end{cases} \tag{7}$$

where $T_1 \leq t,\ t_0 \leq T_2$ *(t_0 fixed),* $L_1 \leq s \leq L_2,\ \lambda(s) \neq 0,\ \varphi$ *is the angle between the principal normal vector field* $N^*(s)$ *of* β *and the surface normal vector field* $n(s,t_0)$ *and* θ *is the angle between the binormal vector field of the curve* α *and the Darboux vector.*

For a better analysis and simplification, we consider the marching-scale functions factored into two factors as

$$u(s,t) = p(s)\,U(t)\,,\quad w(s,t) = r(s)\,W(t)\,,\ v(s,t) = q(s)\,V(t)\,,$$

where $L_1 \leq s \leq L_2$, $T_1 \leq t \leq T_2$ and $p(s)\,,\ q(s)\,,\ r(s)\,,\ U(t)\,,\ V(t)\,,\ W(t)$ are C^1 functions and $p(s)\,,\ q(s)\,,\ r(s)$ are not uniformly zero.

Proposition 3.4 *Let* $\alpha(s)\,,\ L_1 \leq s \leq L_2,$ *be an arclength curve with nonzero curvature and* $\beta(s)\,,\ L_1 \leq s \leq L_2,$ *be its involute.* β *is a line of curvature on the surface* (3) *if*

$$\begin{cases} U(t_0) = W(t_0) = V(t_0) = 0, \\ \theta(s) + \varphi(s) = \textit{constant}, \\ -\kappa(c-s)\,r(s)\,W'(t_0) = \lambda(s)\cos\varphi, \\ \kappa(c-s)\,q(s)\,V'(t_0) = \lambda(s)\sin\varphi, \end{cases} \tag{8}$$

where $L_1 \leq s \leq L_2,\ T_1 \leq t,\ t_0 \leq T_2$ *(t_0 fixed),* $\lambda(s) \neq 0,\ \varphi$ *is the angle between the principal normal vector field* $N^*(s)$ *of* β *and the surface normal vector field* $n(s,t_0)$ *and* θ *is the angle between the binormal vector field of the curve* α *and the Darboux vector.*

4 Examples

Example 1 : Let us consider the arclength circle $\alpha(s) = (\cos s, \sin s, 0)$. Now, it is straight forward to show that

$$\begin{aligned} T(s) &= (-\sin s, \cos s, 0)\,, \\ N(s) &= (-\cos s, -\sin s, 0)\,, \\ B(s) &= (0,0,1)\,, \\ \theta &= 0,\ \kappa = 1,\ \tau = 0. \end{aligned}$$

Letting $c = 0$ in Eqn. (1), we have

$$\beta(s) = (s\sin s + \cos s, \sin s - s\cos s, 0)\,,$$

as an involute of α with Frenet vectors

$$\begin{aligned} T^*(s) &= (-\cos s, -\sin s, 0), \\ N^*(s) &= (\sin s, -\cos s, 0), \\ B^*(s) &= (0, 0, 1). \end{aligned}$$

If we choose $u(s,t) \equiv 0,\ v(s,t) = -\frac{\sqrt{2}}{2}\sin t,\ w(s,t) = \frac{\sqrt{2}}{2}\sin t,\ \lambda(s) = s,\ t_0 = 0$ and $\varphi = \frac{\pi}{4}$ then Eqn. (8) is satisfied and we get the surface

$$\begin{aligned} P_1(s,t) &= \beta(s) + u(s,t)T^* + w(s,t)B^*(s) + v(s,t)N^* \\ &= \left(\cos s + \left(s - \frac{\sqrt{2}}{2}\sin t\right)\sin s, \sin s - \left(s - \frac{\sqrt{2}}{2}\sin t\right)\cos s, \frac{\sqrt{2}}{2}\sin t\right) \end{aligned}$$

$2 \le s \le 5,\ 0 \le t \le 3$, possessing β as an involute line of curvature ($Fig.$ 1).

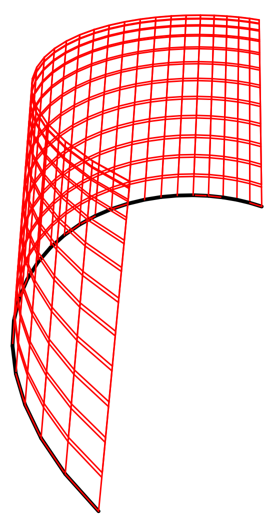

Figure 1: $P_1(s,t)$ as a representative of surfaces and its involute line of curvature β.

Example 2 : Let $\alpha(s) = \left(a_1 \cos\frac{s}{a_3}, a_1 \sin\frac{s}{a_3}, \frac{a_2 s}{a_3}\right)$ be an arc length helix, where $a_1, a_2, a_3 \in \mathbb{R},\ a_1^2 + a_2^2 = a_3^2,\ a_1 > 0$. It is easy show that

$$\begin{aligned} T(s) &= \left(-\frac{a_1}{a_3}\sin\frac{s}{a_3}, \frac{a_1}{a_3}\cos\frac{s}{a_3}, \frac{a_2}{a_3}\right), \\ N(s) &= \left(-\cos\frac{s}{a_3}, -\sin\frac{s}{a_3}, 0\right), \\ B(s) &= \left(\frac{a_2}{a_3}\sin\frac{s}{a_3}, -\frac{a_2}{a_3}\cos\frac{s}{a_3}, \frac{a_1}{a_3}\right), \\ \kappa &= \frac{a_1}{a_3^2},\ \tau = \frac{a_2}{a_3^2},\ \theta = \arctan\frac{a_2}{a_1}. \end{aligned}$$

So we have

$$\begin{aligned} \beta(s) = &\left(a_1\cos\frac{s}{a_3} - \frac{a_1}{a_3}(c-s)\sin\frac{s}{a_3},\right. \\ &\left. a_1\sin\frac{s}{a_3} + \frac{a_1}{a_3}(c-s)\cos\frac{s}{a_3}, \frac{ca_2}{a_3}\right) \end{aligned}$$

as an involute curve of α with Frenet vector fields

$$\begin{aligned} T^*(s) &= \left(-\cos\frac{s}{a_3}, -\sin\frac{s}{a_3}, 0\right), \\ N^*(s) &= sgn(a_3)\left(\sin\frac{s}{a_3}, -\cos\frac{s}{a_3}, 0\right), \\ B^*(s) &= (0, 0, sgn(a_3)). \end{aligned}$$

Choosing $a_1 = \frac{\sqrt{3}}{2},\ a_2 = \frac{1}{2},\ a_3 = 1$ results in $\theta = \frac{\pi}{6}$ and if we let $c = \sqrt{3}$ in Eqn. (1) we get

$$\begin{aligned} \beta(s) = &\left(\frac{\sqrt{3}}{2}\cos s - \frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right)\sin s,\right. \\ &\left.\frac{\sqrt{3}}{2}\sin s + \frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right)\cos s, \frac{\sqrt{3}}{2}\right). \end{aligned}$$

Taking the following marching scale functions $u(s,t) = t,\ w(s,t) = \left(\sqrt{3}+s\right)t,\ v(s,t) = -\frac{1}{\sqrt{3}}\left(\sqrt{3}+s\right)\sin t$, and $\varphi \equiv \frac{\pi}{6},\ t_0 = 0,\ \lambda(s) = s^2 - 3$ Eqn. (8) is satisfied and we obtain

$$\begin{aligned} P_2(s,t) &= \beta(s) + u(s,t)T^*(s) + w(s,t)B^*(s) + v(s,t)N^*(s) \\ &= \left(\left(\frac{\sqrt{3}}{2} - t\right)\cos s - \left[\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) + \frac{1}{\sqrt{3}}\left(\sqrt{3}+s\right)\sin t\right]\sin s,\right. \\ &\left.\left(\frac{\sqrt{3}}{2} - t\right)\sin s + \left[\frac{\sqrt{3}}{2}\left(\sqrt{3}-s\right) + \frac{1}{\sqrt{3}}\left(\sqrt{3}+s\right)\sin t\right]\cos s, \frac{\sqrt{3}}{2} + \left(\sqrt{3}+s\right)t\right), \end{aligned}$$

$0 \le s \le \frac{3}{2},\ -1 \le t \le 1$, as a representative of surfaces interpolating β as an involute line of curvature $(Fig.2)$.

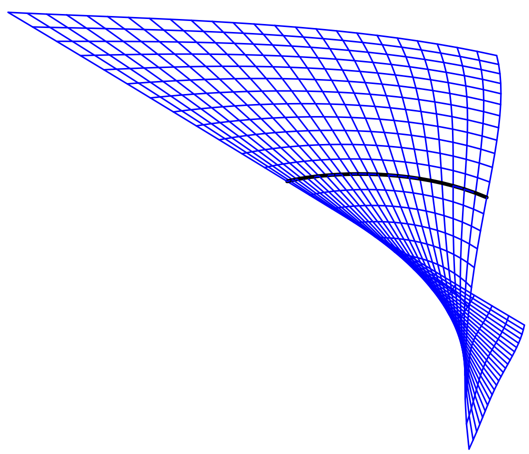

Figure 2: $P_2(s,t)$ as a representative of surfaces and its involute line of curvature β.

If we let $a_1 = a_2 = \frac{1}{2},\ a_3 = \frac{\sqrt{2}}{2}$ then we have $\theta = \frac{\pi}{4}$ and $c = 3$ in formula 1 we obtain the involute of α as

$$\begin{aligned}\beta(s) &= \left(\frac{1}{2}\cos\left(\sqrt{2}s\right) - (3-s)\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}s\right),\right.\\ &\quad \frac{1}{2}\sin\left(\sqrt{2}s\right) + (3-s)\frac{\sqrt{2}}{2}\cos\left(\sqrt{2}s\right),\\ &\quad \left.\frac{3\sqrt{2}}{2}\right).\end{aligned}$$

If we let $u(s,t) = st,\ v(s,t) = st^2,\ w(s,t) = e^2 t,\ \varphi = \pi,\ t_0 = 0,\ \lambda(s) =$

$\frac{\sqrt{3}}{2}e^{s}(s-3)$ then Eqn. (8) is satisfied and we have

$$\begin{aligned} P_3(s,t) &= \beta(s)+u(s,t)T^*(s)+w(s,t)B^*(s)+v(s,t)N^*(s) \\ &= \left(\frac{1}{2}\cos\left(\sqrt{2}s\right)-(3-s)\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}s\right)-st\cos\left(\frac{2\sqrt{3}}{3}s\right)+st^2\sin\left(\frac{2\sqrt{3}}{3}s\right),\right. \\ &\quad \frac{1}{2}\sin\left(\sqrt{2}s\right)+(3-s)\frac{\sqrt{2}}{2}\cos\left(\sqrt{2}s\right)-st\sin\left(\frac{2\sqrt{3}}{3}s\right)-st^2\cos\left(\frac{2\sqrt{3}}{3}s\right), \\ &\quad \left.\frac{3\sqrt{2}}{2}+e^{s}t\right), \end{aligned}$$

$-1\leq s\leq 1,\ 0\leq t\leq 0,9$, as a representative of surfaces accepting β as an involute line of curvature $(Fig.3)$.

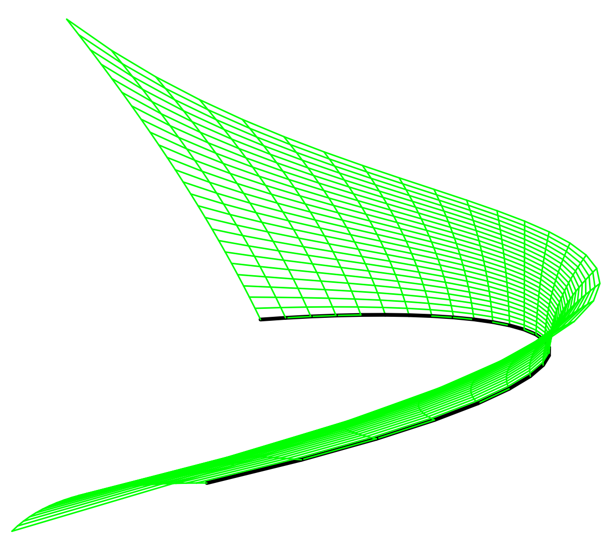

Figure 3: $P_3(s,t)$ as a representative of surfaces and its involute line of curvature β.

For the same curve if we choose $u(s,t)\equiv 0,\ v(s,t)=\ln(1-t),\ w(s,t)=$

$e^s t^2,\ \varphi = \frac{\pi}{2},\ t_0 = 0,\ \lambda(s) = \frac{\sqrt{3}}{2}(s-3)$ then Eqn. (8) is satisfied and we obtain

$$\begin{aligned} P_4(s,t) &= \beta(s) + u(s,t)\,T^*(s) + w(s,t)\,B^*(s) + v(s,t)\,N^*(s) \\ &= \left(\frac{1}{2}\cos\left(\sqrt{2}s\right) - (3-s)\frac{\sqrt{2}}{2}\sin\left(\sqrt{2}s\right) + (\ln(1-t))\sin\left(\frac{2\sqrt{3}}{3}s\right),\right. \\ &\quad \frac{1}{2}\sin\left(\sqrt{2}s\right) + (3-s)\frac{\sqrt{2}}{2}\cos\left(\sqrt{2}s\right) - (\ln(1-t))\cos\left(\frac{2\sqrt{3}}{3}s\right), \\ &\quad \left.\frac{3\sqrt{2}}{2} + e^s t^2\right), \end{aligned}$$

$0 \le s \le 1,\ \ -1 \le t < 1,$ as a representative of the surfaces with an involute line of curvature β $(Fig.\ 4)$.

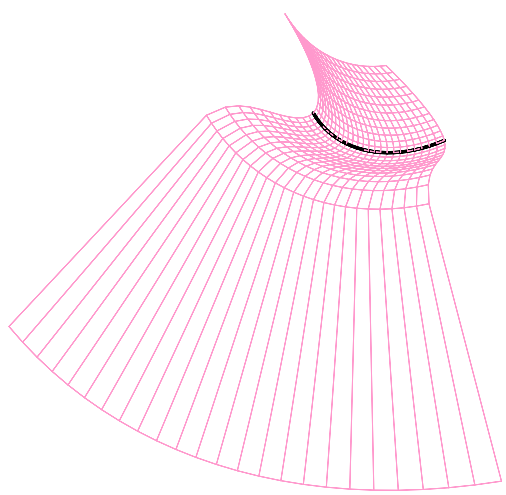

Figure 4: $P_4(s,t)$ as a representative of surfaces and its involute line of curvature β.

5 Open Problem

The construction of surfaces is an interesting problem. There are several methods exist in the literature. In this present paper, we concentrated on the reverse analysis. Given a curve, we obtain surfaces using the involute of the given curve and obtain surfaces accepting the involute as a line of curvature. However, there are so many studies to conduct. Some possible future work deserves to be mentioned. Some constraints, such as mean curvature, Gaussian curvature, minimality may be forced to obtain special surfaces. Same study may be done for implicitly defined surfaces. It is possible to consider other ambient spaces. Minkowski space, Galilean space or Heisenberg space may be considered. Another alternative is to higher the dimension or deal with manifolds.

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