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Limit formulas and a monotonic property related to k-gamma function

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Abstract

In this paper, we mainly present some limit formulas related to ratios of derivatives of the k-gamma function $\Gamma_k(z)$ at their singularities. Furthermore, we also give a monotonic property related to the k-gamma function. Finally, two open problems have been posed.

Keywords: *Limit formulas; K-gamma and polygamma function; Singularity.*

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1 Introduction

The Euler gamma function is defined all positive real numbers x by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function. That is

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}=-\gamma-\frac{1}{x}+\sum_{n=1}^{\infty}\frac{x}{n(n+x)},$$

where $\gamma=0.5772\ldots$ is the Euler-Mascheroni constant. The polygamma functions $\psi^{(m)}(x)$ for $m\in\mathbb{N}$ are defined by

$$\psi^{(m)}(x)=\frac{d^m}{dx^m}\psi(x)=(-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(n+x)^{m+1}},x>0.$$

Origin, history, the complete monotonicity, and inequalities of these special functions may refer to [2, 3, 9].

In 2007, Díaz and Pariguan [1] defined the $k-$analogue of the gamma function for $k>0$ and $x>0$ as

$$\Gamma_k(x)=\int_0^{\infty}t^{x-1}e^{-\frac{t^k}{k}}\,dt=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{x(x+k)\cdots(x+(n-1)k)},$$

where $\lim_{k\to1}\Gamma_k(x)=\Gamma(x)$. Similarly, we may define the $k-$analogue of the digamma and polygamma functions as

$$\psi_k(x)=\frac{d}{dx}\ln\Gamma_k(x)\quad\text{and}\quad\psi_k^{(m)}(x)=\frac{d^m}{dx^m}\psi_k(x).$$

It is well known that the $k-$analogues of the digamma and polygamma functions satisfy the following recursive formula and series identities (See [1, 4, 5, 6])

$$\Gamma_k(k)=1, \tag{1}$$

$$\Gamma_k(x+k)=x\Gamma_k(x),\quad x>0, \tag{2}$$

$$\begin{aligned}\psi_k(x)&=\tfrac{\ln k-\gamma}{k}-\tfrac{1}{x}+\textstyle\sum_{n=1}^{\infty}\tfrac{x}{nk(nk+x)}\\&=-\textstyle\int_0^{\infty}\tfrac{e^{-xt}}{1-e^{-kt}}dt,\end{aligned} \tag{3}$$

and

$$\begin{aligned}\psi_k^{(m)}(x)&=(-1)^{m+1}m!\textstyle\sum_{n=0}^{\infty}\tfrac{1}{(nk+x)^{m+1}}\\&=(-1)^{m+1}\textstyle\int_0^{\infty}\tfrac{1}{1-e^{-kt}}t^me^{-xt}dt.\end{aligned} \tag{4}$$

At present, these functions have been extensively studied.In [13], the second author established a completely monotonic theorem involving the generalized digamma function. In [14], L. Yin, L.-G. Huang, X.-L. Lin and Y.-L. Wang established a concave theorem and some inequalities for k- digamma function. Furthermore, L. Yin, L.-G. Huang, Zh.-M. Song and X.-K. Dou[15] showed several monotonic and concave results related to the generalized digamma and polygamma functions.

In [7] and [8], the limit formulas

$$\lim_{z\to -k}\frac{\Gamma(nz)}{\Gamma(qz)}=(-1)^{(n-q)k}\frac{q}{n}\frac{(qk)!}{(nk)!} \tag{5}$$

and

$$\lim_{z\to -k}\frac{\psi(nz)}{\psi(qz)}=\frac{q}{n} \tag{6}$$

for any non-negative integer k and all positive integers n and q were established by A. Prabhu and H. M. Srivastava. Later, by using explicit formula for the n-th derivative of the cotangent function, F. Qi obtained the following formulas

$$\lim_{z\to -k}\frac{\psi^{(i)}(nz)}{\psi^{(i)}(qz)}=\left(\frac{q}{n}\right)^{i+1} \tag{7}$$

and

$$\lim_{z\to -k}\frac{\Gamma^{(i)}(nz)}{\Gamma^{(i)}(qz)}=(-1)^{(n-q)k}\left(\frac{q}{n}\right)^{i+1}\frac{(qk)!}{(nk)!}. \tag{8}$$

for any non-negative integer k and all positive integers n and q in [10][11] and [12].

It is easily known that the k-gamma function $\Gamma_k(x)$ is single valued and analytic over the entire complex plane, except for the points $z=0,-k,-2k,\cdots$ Motivated by limit formulas (5)- (8), we present some limit formulas related to ratios of derivatives of the k-gamma function $\Gamma_k(z)$ at their singularities. Finally, we also give a monotonic property related to the k-gamma function.

2 Limit formulas at their singularities

Lemma 2.1 *For $k>0$, then*

$$\psi_k(x)=\frac{\ln k}{k}+\frac{\psi(x/k)}{k}. \tag{9}$$

Proof. Taking logarithms and differentiating on both sides of the formula[1]

$$\Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right), \tag{10}$$

we easily obtain the proof.

Remark 2.2 *Considering the formula (4), we obtain that the function $\psi_k(x)$ is strictly increasing on $(0,\infty)$. So the function ψ_k exists an unique root x_k. That is*

$$\psi_k(x_k)=\frac{\ln k+\psi(x_k/k)}{k}=0. \tag{11}$$

Remark 2.3 *Let $k > 0$.*
(i) If $k = 1$, then $x_k = x_0$.
(ii) If $k > 1$, then $x_k < kx_0$.
(iii) If $k < 1$, then $x_k > kx_0$.
Here x_0 satisfies $\psi(x_0) = 0$ with $x_0 = 1.46163\ldots$.

Proof. Using the formula (11), we easily complete the proof.

Remark 2.4 *Using the formula (10), we easily obtain k-analogue of the reflection formula, the duplication formula and the multiplication formula as follows.*

$$\Gamma_k(x)\Gamma_k(k-x) = \frac{\pi}{k\sin\left(\frac{\pi x}{k}\right)}, \tag{12}$$

$$\Gamma_k(2x) = \frac{k^{\frac{1}{2}}2^{\frac{2x}{k}-1}}{\sqrt{\pi}}\Gamma_k(x)\Gamma_k\left(x+\frac{k}{2}\right), \tag{13}$$

$$\Gamma_k(nx) = \frac{k^{\frac{n-1}{2}}n^{\frac{nx}{k}-\frac{1}{2}}}{(2\pi)^{\frac{n-1}{2}}}\Gamma_k(x)\Gamma_k\left(x+\frac{k}{n}\right)\cdots\Gamma_k\left(x+\frac{(n-1)k}{n}\right). \tag{14}$$

Theorem 2.5 *Let $k > 0$. For any non-negative integer λ and all positive integers n, q, we have*

$$\lim_{x\to -k\lambda}\frac{\Gamma_k(nx)}{\Gamma_k(qx)} = k^{\lambda(q-n)}(-1)^{(n-q)\lambda}\frac{q}{n}\frac{(q\lambda)!}{(n\lambda)!}. \tag{15}$$

Proof. Using the formulas (5),(10) and substitution $z = \frac{x}{k}$, we get

$$\begin{aligned}
&\lim_{x\to -k\lambda}\tfrac{\Gamma_k(nx)}{\Gamma_k(qx)} = \lim_{x\to -k\lambda}\frac{k^{\frac{nx}{k}-1}\Gamma\left(\frac{nx}{k}\right)}{k^{\frac{qx}{k}-1}\Gamma\left(\frac{qx}{k}\right)}\\
&= \lim_{x\to -k\lambda}k^{(n-q)z}\tfrac{\Gamma_k(nz)}{\Gamma_k(qz)}\\
&= k^{\lambda(q-n)}(-1)^{(n-q)\lambda}\tfrac{q}{n}\tfrac{(q\lambda)!}{(n\lambda)!}.
\end{aligned}$$

Taking $q = 1, \lambda = 0$ in Theorem2.5, the following Corollary 2.6 holds true.

Corollary 2.6 *Let $k > 0$. For all positive integers n, we have*

$$\lim_{x\to 0}\frac{\Gamma_k(nx)}{\Gamma_k(x)} = \frac{1}{n}. \tag{16}$$

Theorem 2.7 *Let $k > 0$. For any non-negative integer λ, i and all positive integers n, q, we have*

$$\lim_{x\to -k\lambda}\frac{\psi_k^{(i)}(nx)}{\psi_k^{(i)}(qx)} = \left(\frac{q}{n}\right)^{i+1}. \tag{17}$$

Proof. Using the formula (9), we easily get

$$\psi_k^{(i)}(x) = \frac{1}{k^{i+1}}\psi^{(i)}\left(\frac{x}{k}\right). \tag{18}$$

The formulas (7), (18) and substitution $z = \frac{x}{k}$ leads to

$$\lim_{x\to -k\lambda} \frac{\psi_k^{(i)}(nx)}{\psi_k^{(i)}(qx)} = \lim_{x\to -k\lambda} \frac{\psi^{(i)}\left(\frac{nx}{k}\right)}{\psi^{(i)}\left(\frac{qx}{k}\right)}$$
$$= \lim_{z\to -\lambda} \frac{\psi^{(i)}(nz)}{\psi^{(i)}(qz)} = \left(\frac{q}{n}\right)^{i+1}.$$

This completes the proof.

3 A monotonic property related to k-gamma function

Theorem 3.1 *For $0 < k \leq 1$, then the function $H_k(x) = \frac{\Gamma_k'(x+k)}{x}$ is strictly increasing on $(0, \infty)$.*

Proof. Let $0 < x \leq x_k$. Considering Remark 2.2 and Remark 2.3, we get

$$\frac{x}{\Gamma_k(x)} H_k'(x) = x\psi_k(x)\psi_k(k+x) + \psi_k'(k+x)$$
$$= x\psi_k^2(k+x) - \psi_k(k+x) + x\psi_k'(k+x)$$
$$\geq x\psi_k'(k+x) - \psi_k(k+x)$$
$$= x_k\psi_k'(k+x_k) - \psi_k(k+x_k) - \int_x^{x_k} t\psi_k''(k+t)dt$$
$$\geq x_k\psi_k'(k+x_k) - \psi_k(k+x_k)$$
$$= \left(\frac{x_k - k}{k^2}\right)\psi\left(1 + \frac{x_k}{k}\right) - \frac{\ln k}{k}$$
$$> 0.$$

This implies that the function H_k is strictly increasing on $(0, x_k)$. Due to $x \in [x_k, \infty)$, the representation $H_k(x) = \Gamma_k(x)\psi_k(k+x)$ reveals that the function H_k is the product of two functions which are positive and increasing. The proof is complete.

4 Open Problem

Open problem 4.1 Let $k > 0$. Compute the limit

$$\lim_{x\to -k\lambda} \frac{\Gamma_k^{(i)}(nx)}{\Gamma_k^{(i)}(qx)}$$

for any non-negative integer λ, i and all positive integers n, q.

Recently, K. Nantomah, E. Prempeh and S. B. Twum[6] also introduced a new definition of gamma function with two parameters as follows:

$$\Gamma_{p,k}(x) = \frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{(x)_{p,k}}, x > 0$$

where $(x)_{p,k} = x(x+k)(x+2k)...(x+pk)$ and $\lim_{p\to\infty} \Gamma_{p,k}(x) = \Gamma_k(x)$. Naturally, we also pose the following open problem:

Open problem 4.2 Let $p, k > 0$. Compute the limit

$$\lim_{x\to -k\lambda} \frac{\Gamma^{(i)}_{p,k}(nx)}{\Gamma^{(i)}_{p,k}(qx)}$$

for any non-negative integer λ, i and all positive integers n, q satisfying $n\lambda \leq p, q\lambda \leq p$.

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References

[1] R. Díaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol,* Divulg. Mat. **15** 2(2007), 179-192.

[2] B.-N. Guo, F. Qi, *Some properties of the psi and polygamma functions,* Hacet. J. Math. Stat., **39**,2(2010), 219-231.

[3] B.-N. Guo, F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function,* Bull. Korean Math. Soc., **47**,1(2010), 103-111.

[4] K. Nantomah, *Convexity properties and inequalities concerning the (p, k)-gamma functions,* Commun. Fac. Sci. Univ. Ank. Sér. A1. Math. Stat., **66**,2(2017), 130-140.

[5] K. Nantomah, F. Merovci, S. Nasiru, *Some monotonic properties and inequalities for the (p, q)-gamma function,* Kragujevac J. Math., **42**,2(2018), 287-297.

[6] K. Nantomah, E. Prempeh, S. B. Twum, *On a (p, k)-analogue of the gamma function and some associated inequalities,* Moroccan J. Pure Appl. Anal., **2**,2(2016), 79-90.

[7] A. Prabhu, *A property of the gamma function at its singularities*, Available online at [http://front.math.ucdavis.edu/1008.2220]

[8] A. Prabhu, and Srivastava, H. M., *Some limit formulas for the gamma and psi (or digamma) functions at its singularities*, Integral Transforms Spec. Funct., **22** (2011), 587-592

[9] F. Qi, *Bounds for the ratio of two gamma functions,* J. Inequal. Appl., **2010**(2010), 1-84.

[10] F. Qi, *Limit formulas for ratios of polygamma functions at its singularities*, Available online at [http://front.math.ucdavis.edu/1202.2606]

[11] F. Qi, *Limit formulas for ratios of derivatives the gamma and digamma functions at its singularities*, Available online at [http://front.math.ucdavis.edu/1202.4240]

[12] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat, **27** (2013), 601-604

[13] L. Yin, *Complete monotonicity of a function involving the (p; k)-digamma function,* Int. J. Open Problems Compt. Math., **11**, No:2, (2018), 103-108.

[14] L. Yin, L.-G. Huang, X.-L. Lin and Y.-L. Wang, *Monotonicity, concavity, and inequalities related to the generalized digamma function,* Advances in Difference Equations (2018) 2018:246.

[15] L. Yin, L.-G. Huang, Zh.-M. Song and X.-K. Dou, *Some monotonicity properties and inequalities for the generalized digamma and polygamma functions,* Journal of Inequalities and Applications (2018) 2018:249.