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Some Inequalities for Derivatives of the Generalized Wallis' Cosine Formula

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Abstract

In this work, we study some properties and inequalities involving derivatives of a generalized form of the Wallis' cosine (sine) formula. In particular, log-convexity, monotonicity, subadditivity and subhomogeneity properties of the generalized function are discussed.

Keywords: *Generalized Wallis' cosine formula, log-cosine function, log-convex function, subadditivity, inequality.*

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1 Introduction

The classical Euler's Gamma function is usually defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x > 0$. Closely associated with the Gamma function is the digamma or Psi function $\psi(x)$, which is defined as the logarithmic derivative of the Gamma function. That is,

$$\begin{aligned}\psi(x) &= \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \\ &= -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)},\end{aligned}$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = 0.577215664...$ is the Euler-Mascheroni's constant.
The function

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n t\,dt = \int_0^{\frac{\pi}{2}} \sin^n t\,dt = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}, \tag{1}$$

where, $n \in \mathbb{N} = \{1, 2, 3, 4, \dots\}$ is well-known in the literature as the Wallis' cosine (sine) formula. See for instance [1, p. 258], [5], [8] and the related references therin. It may also be defined as

$$I_n = \frac{1}{2}\frac{\Omega_n}{\Omega_{n-1}} = \frac{\pi}{2}W_{\frac{n}{2}} = \frac{1}{2}B\left(\frac{n+1}{2}, \frac{1}{2}\right), \quad n \in \mathbb{N}, \tag{2}$$

where, $\Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^n$, $W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}$ is the Wallis ratio [10], and $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the classical Euler's beta function.

The Wallis' formula and its related functions have been studied intensively by several researchers. Notably are the recent works [4], [9] and [11], where the function was applied to study some properties of a sequence originating from geometric probability for pairs of hyperplanes intersecting with a convex body.
In 1956, Kazarinoff [6] generalized the Wallis' cosine formula as

$$H(x) = \int_0^{\frac{\pi}{2}} \cos^x t\,dt = \int_0^{\frac{\pi}{2}} \sin^x t\,dt = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+1)}, \quad x \in (-1, \infty), \tag{3}$$

where $H(n) = I_n$ for $n \in \mathbb{N}$.
In a recent work [7], the author studied some properties and inequalities involving the generalized function (3). In this paper, our main objective is to derive some inequalities involving derivatives of the generalized function. In doing so, we study some special cases, log-convexity, monotonicity, subadditivity and subhomogeneity properties of the generalized function. We present our results in the following section.

2 Main Results

By differentiating m times of the generalized function (3), we obtain the more generalized form

$$H^{(m)}(x) = \int_0^{\frac{\pi}{2}} (\ln\cos t)^m \cos^x t\,dt = \frac{d^m}{dx^m}\left\{\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+1)}\right\}, \tag{4}$$

for $x \in (-1, \infty)$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $H^{(0)}(x) = H(x)$. In particular, if $x = 0$, then we obtain

$$H^{(m)}(0) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^m \, dt = C_m, \tag{5}$$

which is known in the literature as the log-cosine function [12]. It follows swiftly from (4) that

(i) $H^{(m)}(x)$ is positive and decreasing if $m \in \{2k : k \in \mathbb{N}_0\}$,

(ii) $H^{(m)}(x)$ is negative and increasing if $m \in \{2k+1 : k \in \mathbb{N}_0\}$.

Furthermore, from definition (4), we derive the following few special cases.

$$H^{(0)}(0) = \frac{\pi}{2}, \tag{6}$$

$$H'(0) = \int_0^{\frac{\pi}{2}} \ln \cos t \, dt = -\frac{\pi}{2} \ln 2, \tag{7}$$

$$H'(1) = \int_0^{\frac{\pi}{2}} (\ln \cos t) \cos t \, dt = -1 + \ln 2, \tag{8}$$

$$H'(2) = \int_0^{\frac{\pi}{2}} (\ln \cos t) \cos^2 t \, dt = \frac{\pi}{8} - \frac{1}{4} \ln 2, \tag{9}$$

$$H''(0) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^2 \, dt = \frac{\pi^3}{24} + \frac{\pi}{2} (\ln 2)^2, \tag{10}$$

$$H''(1) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^2 \cos t \, dt = 1 - \frac{\pi^2}{12} + (\ln 2 - 1)^2, \tag{11}$$

$$H''(2) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^2 \cos^2 t \, dt = \frac{\pi}{4} (\ln 2 - 1)^2 + \frac{\pi^3}{48} - \frac{3\pi}{16}, \tag{12}$$

$$H'''(0) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^3 \, dt = -\frac{\pi^3}{8} \ln 2 - \frac{\pi}{2} (\ln 2)^3 - \frac{3\pi}{4} \zeta(3), \tag{13}$$

where $\zeta(x)$ is the Riemann zeta function. Some of these special cases can also be found in [3, p. 531, 582, 585].

Remark 2.1. Some families of these type of integrals have been studied in [2] and as pointed out in that work, these type of integrals have a wide range potential applications in mathematical and physical problems.

In what follows, we present some inequalities involving the function $H^{(m)}(x)$. We start with the following well-known definition.

Definition 2.2. A function $f : I \to \mathbb{R}$ is said to be logarithmically convex or in short log-convex if $\ln f$ is convex on I. That is if

$$\ln f(\alpha x + \beta y) \leq \alpha \ln f(x) + \beta \ln f(y),$$

or equivalently

$$f(\alpha x + \beta y) \leq (f(x))^{\alpha} (f(y))^{\beta},$$

for each $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.

Theorem 2.3. *Let $m, n \in \{2k : k \in \mathbb{N}_0\}$, $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$ and $\frac{m}{a} + \frac{n}{b} \in \mathbb{N}_0$. Then the inequality*

$$H^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a} + \frac{y}{b}\right) \leq \left(H^{(m)}(x)\right)^{\frac{1}{a}} \left(H^{(n)}(y)\right)^{\frac{1}{b}}, \tag{14}$$

is satisfied for $x, y \in (-1, \infty)$.

Proof. The main tool of this proof is the Hölders inequality for integrals. Notice that since $x, y \in (-1, \infty)$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, then $\frac{x}{a} + \frac{y}{b} \in (-1, \infty)$. Then by (4), we obtain

$$\begin{aligned} H^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a} + \frac{y}{b}\right) &= \int_0^{\frac{\pi}{2}} (\ln \cos t)^{\frac{m}{a}+\frac{n}{b}} \cos^{\frac{x}{a}+\frac{y}{b}} t\, dt \\ &= \int_0^{\frac{\pi}{2}} (\ln \cos t)^{\frac{m}{a}} \cos^{\frac{x}{a}} t \cdot (\ln \cos t)^{\frac{n}{b}} \cos^{\frac{y}{b}} t\, dt \\ &\leq \left(\int_0^{\frac{\pi}{2}} (\ln \cos t)^m \cos^x t\, dt\right)^{\frac{1}{a}} \left(\int_0^{\frac{\pi}{2}} (\ln \cos t)^n \cos^y t\, dt\right)^{\frac{1}{b}} \\ &= \left(H^{(m)}(x)\right)^{\frac{1}{a}} \left(H^{(n)}(y)\right)^{\frac{1}{b}}, \end{aligned}$$

which completes the proof. □

Remark 2.4. If $m = n$ in (14), then we obtain

$$H^{(m)}\left(\frac{x}{a} + \frac{y}{b}\right) \leq \left(H^{(m)}(x)\right)^{\frac{1}{a}} \left(H^{(m)}(y)\right)^{\frac{1}{b}}, \tag{15}$$

which is implies that the function $H^{(m)}(x)$ is log-convex on $(-1, \infty)$ if $m \in \{2k : k \in \mathbb{N}_0\}$.

Remark 2.5. If $m = n$ and $a = b = 2$ in (14), then we obtain

$$H^{(m)}\left(\frac{x+y}{2}\right) \leq \sqrt{H^{(m)}(x) H^{(m)}(y)}. \tag{16}$$

Remark 2.6. If $n = m + 2$, $a = b = 2$ and $x = y$ in (14), then we obtain the Turan-type inequality

$$H^{(m)}(x)H^{(m+2)}(x) \geq \left(H^{(m+1)}(x)\right)^2. \tag{17}$$

Corollary 2.7. *The function*

$$h(x) = \frac{H^{(m+1)}(x)}{H^{(m)}(x)}, \tag{18}$$

is increasing on $x \in (-1, \infty)$ *if* $m \in \{2k : k \in \mathbb{N}_0\}$.

Proof. Let $m \in \{2k : k \in \mathbb{N}_0\}$. Then direct differentiation yields

$$h'(x) = \frac{H^{(m+2)}(x)H^{(m)}(x) - \left(H^{(m+1)}(x)\right)^2}{[H^{(m)}(x)]^2} \geq 0,$$

which follows easily from (17). □

Theorem 2.8. *Let* $m \in \{2k : k \in \mathbb{N}_0\}$. *Then the inequality*

$$\left(\frac{H^{(m)}(y)}{H^{(m)}(x)}\right)^{\lambda} \leq \frac{H^{(m)}(\lambda y)}{H^{(m)}(\lambda x)}, \tag{19}$$

holds if either $\lambda \geq 1$ *and* $0 < x \leq y$ *or* $0 < \lambda < 1$ *and* $-1 < x \leq y < 0$. *It reverses if either* $\lambda \geq 1$ *and* $-1 < x \leq y < 0$ *or* $0 < \lambda < 1$ *and* $0 < x \leq y$.

Proof. Let G be defined for $m \in \{2k : k \in \mathbb{N}_0\}$, $\lambda > 0$ and $x \in (-1, \infty)$ by

$$G(x) = \frac{H^{(m)}(\lambda x)}{[H^{(m)}(x)]^{\lambda}}.$$

Then,

$$\frac{G'(x)}{G(x)} = \lambda \left[\frac{H^{(m+1)}(\lambda x)}{H^{(m)}(\lambda x)} - \frac{H^{(m+1)}(x)}{H^{(m)}(x)}\right].$$

If either $\lambda \geq 1$ and $0 < x \leq y$ or $0 < \lambda < 1$ and $-1 < x \leq y < 0$, then we obtain $G'(x) \geq 0$ since $\frac{H^{(m+1)}(x)}{H^{(m)}(x)}$ is increasing. Thus $G(x)$ is increasing. Hence in either case, we have $G(x) \leq G(y)$ which gives (19). Likewise, if either $\lambda \geq 1$ and $-1 < x \leq y < 0$ or $0 < \lambda < 1$ and $0 < x \leq y$, then we obtain $G'(x) \leq 0$ which implies that $G(x)$ is decreasing. Hence we have $G(x) \geq G(y)$ which gives the reverse of (19). □

Theorem 2.9. *Let* $m, u \in \{2k : k \in \mathbb{N}_0\}$ *and* $m \geq u$. *Then the Turan-type inequality*

$$\exp\left\{H^{(m-u)}(x)\right\} \cdot \exp\left\{H^{(m+u)}(x)\right\} \geq \left[\exp\left\{H^{(m)}(x)\right\}\right]^2, \tag{20}$$

holds for $x \in (-1, \infty)$.

Proof. Let $m, u \in \{2k : k \in \mathbb{N}_0\}$ and $m \geq u$. Then by using (4), we obtain the following estimation.

$$\begin{aligned}
&\frac{H^{(m-u)}(x)}{2} + \frac{H^{(m+u)}(x)}{2} - H^{(m)}(x) \\
&= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} (\ln \cos t)^{m-u} \cos^x t\, dt + \int_0^{\frac{\pi}{2}} (\ln \cos t)^{m+u} \cos^x t\, dt\right] \\
&\quad - \int_0^{\frac{\pi}{2}} (\ln \cos t)^{m} \cos^x t\, dt \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[\frac{1}{(\ln \cos t)^u} + (\ln \cos t)^u + 2\right] (\ln \cos t)^{m} \cos^x t\, dt \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left[1 + (\ln \cos t)^u\right]^2 (\ln \cos t)^{m-u} \cos^x t\, dt \\
&\geq 0.
\end{aligned}$$

Thus

$$\frac{H^{(m-u)}(x)}{2} + \frac{H^{(m+u)}(x)}{2} \geq H^{(m)}(x),$$

and by taking exponents, we obtain the result (20). □

Theorem 2.10. *Let $m \in \{2k : k \in \mathbb{N}_0\}$. Then the inequality*

$$H^{(m)}(x+y) \leq H^{(m)}(x) + H^{(m)}(y), \tag{21}$$

holds for $x, y \in [0, \infty)$. In other words, $H^{(m)}(x)$ is subadditive on $[0, \infty)$ if $m \in \{2k : k \in \mathbb{N}_0\}$.

Proof. Let $a > 1$, $b > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then by the Hölder's inequality, we obtain

$$\begin{aligned}
H^{(m)}(x+y) &= \int_0^{\frac{\pi}{2}} (\ln \cos t)^{\frac{m}{a}+\frac{m}{b}} \cos^{x+y} t\, dt \\
&= \int_0^{\frac{\pi}{2}} (\ln \cos t)^{\frac{m}{a}} \cos^{x} t \cdot (\ln \cos t)^{\frac{m}{b}} \cos^{y} t\, dt \\
&\leq \left(\int_0^{\frac{\pi}{2}} (\ln \cos t)^{m} \cos^{ax} t\, dt\right)^{\frac{1}{a}} \left(\int_0^{\frac{\pi}{2}} (\ln \cos t)^{m} \cos^{by} t\, dt\right)^{\frac{1}{b}} \\
&= \left(H^{(m)}(ax)\right)^{\frac{1}{a}} \left(H^{(m)}(by)\right)^{\frac{1}{b}}.
\end{aligned}$$

That is

$$H^{(m)}(x+y) \leq \left(H^{(m)}(ax)\right)^{\frac{1}{a}} \left(H^{(m)}(by)\right)^{\frac{1}{b}}. \tag{22}$$

Then by the Young's inequality:

$$u^{\frac{1}{a}}v^{\frac{1}{b}} \leq \frac{u}{a} + \frac{v}{b},$$

where $u, v \geq 0$, $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$, we obtain

$$\left(H^{(m)}(ax)\right)^{\frac{1}{a}} \left(H^{(m)}(by)\right)^{\frac{1}{b}} \leq \frac{H^{(m)}(ax)}{a} + \frac{H^{(m)}(by)}{b}. \tag{23}$$

Furthermore, since $a > 1$, $b > 1$ and $H^{(m)}(x)$ is decreasing for $m \in \{2k : k \in \mathbb{N}_0\}$, we have

$$\frac{H^{(m)}(ax)}{a} + \frac{H^{(m)}(by)}{b} \leq H^{(m)}(x) + H^{(m)}(y). \tag{24}$$

Finally, by combining (22), (23) and (24), we obtain the result (21). $\square$

Remark 2.11. If $x = y$ in (21), then we obtain

$$H^{(m)}(2x) \leq 2H^{(m)}(x). \tag{25}$$

By repeated applications of (21) and (25), we obtain

$$H^{(m)}(nx) \leq nH^{(m)}(x), \quad n \in \mathbb{N}, \tag{26}$$

which implies that $H^{(m)}(x)$ is $\mathbb{N}$-subhomogeneous for $m \in \{2k : k \in \mathbb{N}_0\}$.

Remark 2.12. Note that $H^{(0)}(n) = I_n$ for $n \in \mathbb{N}$. Then as a special case, by letting $m = 0$, $x = r \in \mathbb{N}$ and $y = s \in \mathbb{N}$ in (21), we obtain

$$I_{r+s} \leq I_r + I_s,$$

which implies that the Wallis' sequence I_n is subadditive.

Theorem 2.13. *Let $m \in \{2k : k \in \mathbb{N}_0\}$. Then the inequality*

$$H^{(m)}(x)H^{(m)}(y) \leq C_m H^{(m)}(x + y), \tag{27}$$

holds for $x, y \in [0, \infty)$, where $C_m = \int_0^{\frac{\pi}{2}} (\ln \cos t)^m \, dt$.

Proof. Let T be defined for $m \in \{2k : k \in \mathbb{N}_0\}$ by

$$T(x, y) = \frac{H^{(m)}(x)H^{(m)}(y)}{H^{(m)}(x + y)}, \quad x, y \in [0, \infty),$$

and let $\delta(x, y) = \ln T(x, y)$. With no loss of generality, let y be fixed. Then,

$$\delta'(x, y) = \frac{H^{(m+1)}(x)}{H^{(m)}(x)} - \frac{H^{(m+1)}(x + y)}{H^{(m)}(x + y)} \leq 0,$$

since $\frac{H^{(m+1)}(x)}{H^{(m)}(x)}$ is increasing (see Corollary 2.7). Thus, $\delta(x, y)$ is decreasing and consequently, $T(x, y)$ is also decreasing. Then for $x \geq 0$, we obtain

$$\frac{H^{(m)}(x)H^{(m)}(y)}{H^{(m)}(x + y)} \leq H^{(m)}(0) = C_m,$$

which gives the result (27). $\square$

3 Open Problem

Is there an explicit expression for the function

$$C_m = \int_0^{\frac{\pi}{2}} (\ln \cos t)^m \, dt, \quad m \in \mathbb{N}_0,$$

in terms of other special functions?

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