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Transcendence criteria with negative base

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Abstract

The aim of this paper is to prove that the strings of 0 ***in the*** $(-\beta)$***-expansion of*** $\ell_\beta = \frac{-\beta}{\beta+1}$ ***exhibit a lacunary bounded when*** β ***is an algebraic number greater than*** 1***. This result provides in a naturel way a transcendence criteria with numeration system in negative base..***

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1 Introduction

There exist many ways to represent real numbers. Besides, expansions in integer negative base $-b$, where $b > 2$, seem to have been introduced by Grunwald in [3], and rediscovered by several authors, see the historical comments given by Knuth [5]. The choice of a negative base $-b$ and of the alphabet $\{0, \cdots, b-1\}$ is interesting, because it provides a signless representation for every number (positive or negative). In this case it is easy to distinguish the sequences representing a positive integer from the ones representing a negative integer: denoting $(x.)_{-b} = \sum_{i=0}^{k} x_i(-b)^i$ for any $x = x_k \cdots x_1x_0$ in $\{0, \cdots, b-1\}^*$ with no leading 0's, we have $\mathbb{N} = \{(x.)_{-b} \mid |x| \ is\ odd\}$.

Recently, Ito and Sadahiro [4] suggested to study positional systems with no integer negative bases $-\beta$, where $\beta > 1$. They have provided a condition for admissibility of digit strings as $(-\beta)$-expansions and shown some properties of the dynamical system connected to $(-\beta)$- numeration. Representation of real numbers in such a system is defined using the transformation $T_{-\beta} : [\ell_\beta, r_\beta) \longmapsto$

$[\ell_\beta, r_\beta)$, where $\ell_\beta = -\frac{\beta}{\beta+1}$, $r_\beta = \ell_\beta + 1 = \frac{1}{\beta+1}$. the transformatio $T_{-\beta}$ given by the prescription

$$T_{-\beta}(x) := -\beta x - [-\beta x - l_\beta].$$

Every real $x \in I_\beta := [\ell_\beta, r_\beta)$ is a sum of the infinite series

$$x = \sum_{i=1}^{+\infty} \frac{x_i}{(-\beta)^i}, \qquad \text{where } x_i = [-\beta T_{-\beta}^{i-1}(x) - l_\beta] \text{ for } i = 1, 2, 3 \ldots \tag{1}$$

Directly from the definition of the transformation $T_{-\beta}$ we can derive that the digits x_i take values in the alphabet $A_\beta = \{0, 1, \cdots, [\beta]\}$ for $i = 1, 2, \cdots$. The expression of x in the form (3) is called the $(-\beta)$-expansion of x. The number x is thus represented by the infinite word $d_{-\beta}(x) := x_1 x_2 \cdots \in A_\beta^{\mathbb{N}}$.
In [4] it is suggested to find the expansion of a number $x \notin [l_\beta, r_\beta)$ by dividing it by a suitable power of $(-\beta)$ so that $y := (-\beta)^{-k} x \in [l_\beta, r_\beta)$, finding the expansion of y and multiplying it back by $(-\beta)^k$. The expression for x provided by such procedure, however, depends on chosen k, so the prescription must be modified, in order to give a unique $(-\beta)$-expansion for every real x. The advantage of this numeration system with negative base stems from the fact that both positive and negative real numbers can be represented with non-negative digits. Thereby, in [1] we give a property that allows us to know the sign of a certain real number in $\mathbb{R}$ after his $(-\beta)$-expansion.

Lemma 1.1 *Let $\beta > 1$ such that $d_{-\beta}(x) = a_{-l} a_{-l+1} a_{-l+2} \ldots a_0 . a_1 a_2 \ldots$ for every real number x and $s = \inf_{i \geq -l} \{i; a_i \neq 0\}$. Then $sign(x) = sign(-1)^s$.*

From the definition of the transformation $T_{-\beta}$, Ito and Sadahiro has provided a criteria to decide whether an infinite word $A^{\mathbb{N}}$ belongs to the set of $(-\beta)$-expansions. We give this criterion:

Proposition 1.2 *An integer sequence $(x_1, x_2, \ldots)$ represents a $(-\beta)$-expansion of some $x \in I_\beta$ if and only if $T_{-\beta}^i(x) \in I_\beta$ for all $i \geq 1$.*

To make the last Proposition more explicit, Ito and Sadahiro introduce in [4] the ordering on the set of infinite words over the alphabet A_β which would correspond to the ordering of real numbers is the so-called alternate ordering: We say that $x_1 x_2 x_3 \ldots \prec_{alt} y_1 y_2 y_3 \ldots$ if for the minimal index j such that $x_j \neq y_j$ it holds that $x_j (-1)^j < y_j (-1)^j$. In this notation, we can write for arbitrary $x, y \in I_\beta$ that

$$x \leq y \quad \Leftrightarrow \quad d_{-\beta}(x) \preceq_{alt} d_{-\beta}(y)$$

In their paper, In order to describe the digit strings that are admissible as $(-\beta)$-expansions, one defines the so-called a $(-\beta)$ represantation of r_β denoted $d^*_{-\beta}(r_\beta)$ by:

$$d^*_{-\beta}(r_\beta) = \begin{cases} (0b_1b_2\cdots b_{q-1}b_q-1)^w & if\ d_{-\beta}(\ell_\beta)=(b_1b_2\cdots b_q)^w \ \ and \text{ q } odd, \\ d_{-\beta}(r_\beta) = 0d_{-\beta}(\ell_\beta) & otherwise. \end{cases}$$

(As usual, the notation a^w stands for infinite repetition of the string a.)
They have shown that a digit string $x_k x_{k-1}\cdots$ is admissible if and only if each of its suffixes satisfies

$$\forall\ i\in\mathbb{N}^*,\ d_{-\beta}(\ell_\beta) \preceq_{alt} (x_i x_{i+1} x_{i+2}\cdots) \prec_{alt} d^*_{-\beta}(r_\beta).$$

In particular, in [8] Steiner proved that a sequence $b_1b_2\cdots$ is the $(-\beta)$-expansions of ℓ_β if and only if it satisfies the following condition:
i) $b_1b_2\cdots \preceq_{alt} b_ib_{i+1}b_{i+2}\cdots$.
ii) $b_1b_2\cdots \prec_{alt} u_1u_2\cdots = 1001110010010011100 11\cdots$,
where $u_1u_2\cdots$ is the sequence starting with $\varphi^n(1)$ for all $n\geq 0$ with φ being the morphism of words on the alphabet $\{0,1\}$ defined by $\varphi(1)=100,\ \varphi(0)=1$.
iii) $b_1b_2\cdots \notin \{b_1\cdots b_k,\ b_1\cdots b_{k-1}(b_k-1)0\}^w \setminus \{(b_1\cdots b_k)^w\}$ for all $k\geq 1$, with $(b_1\cdots b_k)^w \prec u_1u_2\cdots$.
iv) $b_1b_2\cdots \notin \{b_1\cdots b_k0,\ b_1\cdots b_{k-1}(b_k+1)\}^w \setminus \{(b_1\cdots b_k)^w\}$ for all $k\geq 1$, with $(b_1\cdots b_{k-1}(b_k+1))^w \prec u_1u_2\cdots$.

The $(-\beta)$-expansion of ℓ_β, $d_{-\beta}(\ell_\beta)$ is important since it yields a lot of information on the classiffication of algebraic numbers $\beta>1$. In [7], Masakova, Pelantova, Ito and Sadahiro have called these bases Ito-Sadahiro numbers and they have proved that an Ito-Sadahiro number is an algebraic integer. Frougny and Lai show in [2] that if β is an Ito-Sadahiro number, then β is a Pisot number (an algebraic integer whose conjugates have modulus <1). It is proved in [7, 2] that an Ito-Sadahiro number is a Perron number (an algebraic integer whose conjugates have modulus $<[\beta]$).

Now, we are interested if the sequence of $(-\beta)$-expansion of ℓ_β is infinite. In this context, we give in this paper a new Theorem on the gaps (strings of 0's) in $d_{-\beta}(\ell_\beta)$ for algebraic numbers $\beta>1$.

2 Main results

Theorem 2.1 *Let $\beta>1$ be an algebraic number with minimal polynomial*

$$P_\beta(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_0$$

and $M(\beta)$ be its Mahler measure defined by

$$M(\beta) = |a_d|\prod_{i=0}^{d-1} max\{1, |\beta_i|\},$$

where $\beta=\beta_0,\beta_1,\cdots,\beta_{d-1}$ are the complex conjugates of β.
Denote by $d_{-\beta}(\ell_\beta):=0.t_1t_2t_3\cdots$, the $(-\beta)$-expansion of ℓ_β. Assume that $d_{-\beta}(\ell_\beta)$ is infinite and gappy in the following sense: there exist two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that

$$1=s_0\leq m_1<s_1\leq m_2<s_2\leq\cdots\leq m_n<s_n\leq m_{n+1}<s_{n+1}\leq\cdots$$

with $(s_n-m_n)\geq 2$, $t_{m_n}\neq 0$, $t_{s_n}\neq 0$ and $t_i=0$ if $m_n<i<s_n$ for all $n\geq 1$. Then,

$$\limsup_{n\to+\infty}\frac{s_n}{m_n}\leq\frac{logM(\beta)+(d-1)Log|a_d|}{Log\beta}.$$

Corollary 2.2 *If β is a Pisot or Salem number and the $(-\beta)$-expansion of ℓ_β satisfies the same conditions of Theorem 2.1, then*

$$\limsup_{n\to+\infty}\frac{s_n}{m_n}\leq 1.$$

Before giving the proof of the theorem 2.1, we present the following lemmas

Lemma 2.3 *(Classical theorem on symmetric polynomials)*
Let $Q\in\mathbb{Z}[X]$ and $P(y_1,y_2,\ldots,y_n)=Q(y_1)Q(y_2)\ldots Q(y_n)$. Then it exists a polynomial T of n variables with coefficients in $\mathbb{Z}$ such as

$$P(y_1,y_2,\ldots,y_n)=T(\sigma_1,\sigma_2,\ldots,\sigma_n)$$

where:

$$\begin{cases}\sigma_1=\sum_{i=1}^{n}y_i\\ \sigma_2=\sum_{1\leq i<j\leq n}y_iy_j\\ \sigma_3=\sum_{1\leq i<j<k\leq n}y_iy_jy_k\\ \vdots\\ \sigma_n=y_1y_2\ldots y_n\end{cases}$$

Moreover, we note that the total degree of T is less than or equal to the degree of Q (as polynomial in y)

Lemma 2.4 *Let β be an algebraic number with minimal polynomial $P_\beta(x)=a_dx^d+a_{d-1}x^{d-1}+\cdots+a_0$ where $a_i\in\mathbb{Z}$, for all $0\leq i\leq d$.*
Let $Q(x)=b_nx^n+b_{n-1}x^{n-1}+\cdots+b_0$ where $b_i\in\mathbb{Z}$, for all $0\leq i\leq n$, such that $Q(\beta)\neq 0$.
Then, we have

$$|Q(\beta)|\geq\frac{\beta^n}{(H(Q))^{d-1}(n+1)^{d-1}(M(\beta))^n|a_d|^{n(d-1)+d}},$$

where $H(Q)$ is the height of Q defined by $H(Q)=\max_{0\leq i\leq n}|b_i|$.

Proof 2.5 *Let* $\beta_o = \beta, \dots, \beta_{d-1}$ *the conjugates of* β *and* $P_\beta(x) = a_d x^d + a_{d-1}x^{d-1} + \dots + a_0$ *where* $a_i \in \mathbb{Z}$, *for all* $0 \le i \le d.$, *the minimal polynomial of* β, *then* $a_d\beta^d = \sum_{i=0}^{d-1} -a_i\beta^i$, *therefore for all* $k \in \mathbb{N}^*$;

$$a_d^k \beta^{d+k-1} = \sum_{i=0}^{d-1} \alpha_{k,i}\beta^i, \quad \alpha_{k,i} \in \mathbb{Z}.$$

This gives that for all $s' \ge s$:

$$a_d^{s'} \beta^s = \sum_{i=0}^{d-1} c_{k,i}\beta^i, \quad c_{k,i} \in \mathbb{Z}.$$

So, $a_d^n Q(\beta) = \sum_{i=0}^{d-1} \theta_{k,i}\beta^i, \quad \theta_{k,i} \in \mathbb{Z}.$
Therefore:

$$a_d^n Q(\beta_j) = \sum_{i=0}^{d-1} e_{k,i}\beta_j^i, \quad e_{k,i} \in \mathbb{Z}, \quad \forall\ 0 \le j \le d-1.$$

Hence, we obtain $a_d^{nd}\prod_{j=0}^{d-1} Q(\beta_j) = \prod_{j=0}^{d-1}(\sum_{i=0}^{d-1} e_{k,i}\beta_j^i)$. *By Lemma 2.3, it exists a polynomial* T *of* d *variables with coefficients in* $\mathbb{Z}$, *such that*

$$\prod_{j=0}^{d-1}(\sum_{i=0}^{d-1} e_{k,i}\beta_j^i) = T(\sigma_1, \sigma_2, \dots, \sigma_d),$$

where

$$\sigma_j = \sum_{0 \le k_1 < k_2 < \dots < k_j \le d-1} \beta_{k_1}\beta_{k_2}\dots\beta_{k_j}, \quad 1 \le j \le d,$$

and the total degree of T *is lower than or equal to the degree of* d. *Which gives that* $a_d^d a_d^{nd}\prod_{j=0}^{d-1} Q(\beta_j) \in \mathbb{Z}^*$.

Consequently, $\mid \prod_{j=0}^{d-1} Q(\beta_j) \mid \ge \dfrac{1}{|a_d|^{nd+d}}$.
$\mid Q(\beta_j) \mid < H(Q)(n+1)\sup(|\beta_j|^n, 1)$, *so*

$$\begin{aligned} \mid \prod_{j=1}^{d-1} Q(\beta_j) \mid &\le H(Q)^{d-1}\prod_{j=1}^{d-1}(n+1)\sup(|\beta_j|^n, 1), \\ &\le \frac{(n+1)^{d-1}H(Q)^{d-1}(M(\beta))^n}{|a_d|^n\beta^n}. \end{aligned}$$

Hence, we obtain

$$| Q(\beta) |\geq \frac{1}{|\prod_{j=1}^{d-1} Q(\beta_j) | \, |a_d|^{nd+d}} \geq \frac{\beta^n}{(H(Q))^{d-1}(n+1)^{d-1}(M(\beta))^n |a_d|^{n(d-1)+d}}.$$

Proof of Theorem 2.1 : By contradiction, we assume that

$$\limsup_{n\to+\infty} \frac{s_n}{m_n} > \frac{logM(\beta)+(d-1)Log|a_d|}{Log\beta}. \tag{2}$$

Set $S_n(z) = (-1)^{m_n+1}z^{m_n+1}+(-1)^{m_n}t_1z^{m_n}+\sum_{i=1}^{m_n-1}(-1)^{m_n-i}(t_{i+1}-t_i)z^{m_n-i}-t_{m_n}$
a polynomial of integer coefficients of degree m_n+1.
We have $\ell_\beta = \sum_{i=1}^{\infty} \frac{t_i}{(-\beta)^i}$ i.e

$$\frac{-\beta}{\beta+1} = \frac{t_1}{(-\beta)} + \cdots + \frac{t_{m_n}}{(-\beta)^{m_n}} + \frac{t_{s_n}}{(-\beta)^{s_n}} + \cdots .$$

Multiplying the two sides by $(-\beta)^{m_n}$, we obtain

$$(-\beta)^{m_n+1} = (\beta+1)[t_1(-\beta)^{m_n-1}+\cdots+t_{m_n}] + (\beta+1)[\frac{t_{s_n}}{(-\beta)^{s_n-m_n}}+\cdots],$$

which implies that

$$(-\beta)^{m_n+1} - (\beta+1)[t_1(-\beta)^{m_n-1}+\cdots+t_{m_n}] = (\beta+1)\sum_{i=s_n}^{+\infty}\frac{t_i}{(-\beta)^{i-m_n}},$$

thereby

$$(-\beta)^{m_n+1}+t_1(-\beta)^{m_n}+(t_2-t_1)(-\beta)^{m_n-1}+\cdots+(t_{m_n}-t_{m_n-1})(-\beta)-t_{m_n} = (\beta+1)\sum_{i=s_n}^{+\infty}\frac{t_i}{(-\beta)^{i-m_n}}.$$

Hence

$$\begin{aligned}|S_n(\beta)| &= |(\beta+1)\sum_{i=s_n}^{+\infty}\frac{t_i}{(-\beta)^{i-m_n}}|,\\ &= (\beta+1)(\frac{t_{s_n}}{(-\beta)^{s_n-m_n}}+\cdots)|,\\ &\leq \frac{1}{\beta^{s_n-m_n}}(\beta+1)\frac{\beta[\beta]}{\beta-1}. \end{aligned} \tag{3}$$

Recall that the minimal polynomial of $\beta=\beta_0$ is

$$P_\beta(x) = \sum_{i=0}^{d} a_i x^i = a_d\prod_{i=0}^{d-1}(x-\beta_i)$$

with $\beta_1, \cdots, \beta_{d-1}$ are their conjugates and $S_n(z) \in \mathbb{Z}[X]$. By lemma 2.4, we have

$$|S_n(\beta)| \geq \frac{\beta^{m_n+1}}{[\beta]^{d-1}(m_n+2)^{d-1}(M(\beta))^{m_n+1}|a_d|^{(m_n+1)(d-1)+d}}. \tag{4}$$

Combining 3 and 4 we obtain:

$$\begin{aligned}
\frac{\beta^{m_n+1}}{[\beta]^{d-1}(m_n+2)^{d-1}(M(\beta))^{m_n+1}|a_d|^{(m_n+1)(d-1)+d}} &\leq \frac{1}{\beta^{s_n-m_n}}(\beta+1)\frac{\beta[\beta]}{\beta-1}, \\
\frac{\beta^{s_n}}{|a_d|^d[\beta]^{d-1}(m_n+2)^{d-1}(|a_d|^{d-1}M(\beta))^{m_n+1}} &\leq \frac{(\beta+1)}{(\beta-1)}[\beta],
\end{aligned}$$

$$\frac{1}{|a_d|^{2d-1}[\beta]^{d-1}(m_n+2)^{d-1}M(\beta)}\left(\frac{\beta^{\frac{s_n}{m_n}}}{|a_d|^{d-1}M(\beta)}\right)^{m_n} \leq [\beta]\frac{\beta+1}{\beta-1}. \tag{5}$$

Denote for $n \geq 1$;

$$u_n =: \frac{1}{|a_d|^{2d-1}[\beta]^{d-1}(m_n+2)^{d-1}M(\beta)}\left(\frac{\beta^{\frac{s_n}{m_n}}}{|a_d|^{d-1}M(\beta)}\right)^{m_n}.$$

Hence, we get that the sequence u_n is bounded. So by the inequality 2 there exists a sequence of integers (n_i) which tends to infinity and an integer i_0 such that

$$\frac{s_{n_i}}{m_{n_i}} > \frac{log(|a_d|^{d-1}M(\beta))}{Log\beta} \qquad \text{for all } i \geq i_0.$$

Thereby returning that for $i \geq i_0$;

$$\frac{\beta^{\frac{s_{n_i}}{m_{n_i}}}}{|a_d|^{d-1}M(\beta)} > \frac{\beta^{\frac{log(|a_d|^{d-1}M(\beta))}{Log\beta}}}{|a_d|^{d-1}M(\beta)} = 1.$$

This implies that the subsequence u_{n_i} tends exponentially to infinity when i tends to infinity, which contradicts the bounded of sequence u_n.

Corollary 2.6 *Let $\beta > 1$ be a real number such that the $(-\beta)$-expansion of ℓ_β is the form*

$$d_{-\beta}(\ell_\beta) := 0.t_1t_2t_3\cdots, \quad \textit{with} \quad t_i \in A_\beta := \{0;1;2;\cdots;[\beta]\},$$

Assume that $d_{-\beta}(\ell_\beta)$ is infinite and gappy in the following sense: There exist two sequences $(m_n)_{n\geq 1}$ and $(s_n)_{n\geq 0}$ such that

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \cdots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \cdots$$

with $(s_n - m_n) \geq 2$, $t_{m_n} \neq 0$, $t_{s_n} \neq 0$ and $t_i = 0$ if $m_n < i < s_n$ for all $n \geq 1$. If $\limsup_{n\to+\infty} \frac{s_n}{m_n} = +\infty$, then β is a transcendental number.

3 Application

Thank to *i*), *ii*), *iii*), and *iv*) there exist $\beta > 1$ such that the $(-\beta)$-expansion of ℓ_β is in the form:

$$d_{-\beta}(\ell_\beta) = 1\underbrace{0}_{\lambda_1}1\underbrace{000\cdots}_{\lambda_2}1\underbrace{000\cdots}_{\lambda_3}1\underbrace{000\cdots}_{\lambda_4}1\cdots,$$

with $(\lambda_n)_{n\geq 1} = n^{n^2}$. A simple computation prove that for this example we have:

$$m_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n + n + 1$$

$$\text{and } s_n = m_n + \lambda_{n+1} + 1,$$

then $\limsup_{n\to+\infty} \frac{s_n}{m_n} = +\infty$.
According to corollary 2.6, β is necessarily a transcendental number.

4 Open Problem

The main result is it still true if we replace the string of "0" by a string of any integer $a > 0$.

References

[1] S. Dammak and M. Hbaib, *Number systems with negative bases*, Acta Math Hunger. 2, (2014), 457-483.

[2] Ch. Frougny et A. Chiara Lai, *Negative bases and automata*, Discr. Math. Theor. Comp. Sci, 13, (2011), 75−94.

[3] V. Grunwald, *Intorno allaritmetica dei sistemi numerici a base negativa con particolare riguardo al sistema numerico a base negativo-decimale per lo studio delle sue analogie collaritmetica ordinaria (decimale)*, Giornale di Matematiche di Battaglini 367, (1885) 203-221.

[4] S. Ito and T. Sadahiro, *Bêta-expansions with negative bases*, Integers. 9, (2009), 239−259.

[5] D.E. Knuth, *The Art of Computer Programming*, Seminumerical Algorithms .2, 2nd ed. Addison- Wesley (1988).

[6] Z. Masakova, E. Pelantova and T. Vavra, *Arithmetics in number systems with a negative base*, Theor. Comp. Sci. 12, (2011), 835-845.

[7] Z. Masáková, E. Pelantová, S. Ito and T. Sadahiro, *numbers vs. Parry numbers*, Acta Polytechnica, 51, (2011), 59−64.

[8] W. Steiner, *Digital expansion with negative real bases*, Acta Math Hunger. 2, (2013), 106-119.