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A generalization of the Nielsen's β-function

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Abstract

In this paper, we introduce a p***-generalization of the Nielsen's*** β***-function. We further study among other things, some properties such as convexity, monotonicity and inequalities of the new function. In the end, we pose an open problem.***

Keywords: *Nielsen's β-function, p-generalization, p-Gamma function, convolution theorem for Laplace transforms, completely monotonic.*
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1 Introduction

The Nielsen's β-function may be defined by any of the following equivalent forms (see [2], [3], [8], [11]).

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad x>0, \tag{1}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt, \quad x>0, \tag{2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x>0, \tag{3}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x>0, \tag{4}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler's Gamma function. It is known to satisfy the properties:

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{5}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}. \tag{6}$$

Lately, this special function has been studied in diverse ways. For instance, in [8], the author investigated some properties and inequalities of the function. Also, in [9], the function was applied to study some monotonicity and convexity properties and some inequalities involving a generalized form of the Wallis' cosine formula. Then in [10], the author proved some monotonicity and convexity properties of the function. In this paper, we continue the investigation by establishing a p-generalization of this special function. In the meantime, we recall the following definitions concerning the p-analogue of the Gamma function. We shall use the notations $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

The p-analogue (also known as p-extension or p-deformation) of the Gamma function is defined for $p \in \mathbb{N}$ and $x > 0$ as [1], [12]

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\dots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\dots(1+\frac{x}{p})} \tag{7}$$

$$= \int_0^p \left(1-\frac{t}{p}\right)^p t^{x-1}\,dt \tag{8}$$

where $\lim_{p\to\infty}\Gamma_p(x) = \Gamma(x)$. It satisfies the identities [5]

$$\Gamma_p(x+1) = \frac{px}{x+p+1}\Gamma_p(x),$$

$$\Gamma_p(1) = \frac{p}{p+1}.$$

The p-analogue of the digamma functions is defined for $x > 0$ as [6]

$$\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x) = \ln p - \sum_{n=0}^{p}\frac{1}{n+x}, \tag{9}$$

$$= \ln p - \int_0^\infty \frac{1-e^{-(p+1)t}}{1-e^{-t}}e^{-xt}\,dt, \tag{10}$$

and satisfies the relation [5]

$$\psi_p(x+1) = \frac{1}{x} - \frac{1}{x+p+1} + \psi_p(x). \tag{11}$$

Also, it is well known in the literature that the integral

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt}\,dt \tag{12}$$

holds for $x > 0$ and $m \in \mathbb{N}_0$.

2 A p-Generalization of Nielsen's β-function

In this section, we introduce a p-generalization of the Nielsen's β-function and further study some of its properties.

Definition 2.1. The p-generalization of the Nielsen's β-function is defined for $p \in \mathbb{N}$ as

$$\beta_p(x) = \frac{1}{2}\left\{\psi_p\left(\frac{x+1}{2}\right) - \psi_p\left(\frac{x}{2}\right)\right\}, \quad x > 0, \tag{13}$$

$$= \sum_{n=0}^{p}\left(\frac{1}{2n+x} - \frac{1}{2n+x+1}\right) \quad x > 0, \tag{14}$$

$$= \int_0^\infty \frac{1-e^{-2(p+1)t}}{1+e^{-t}} e^{-xt}\, dt, \quad x > 0, \tag{15}$$

$$= \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{x-1}\, dt, \quad x > 0, \tag{16}$$

where $\beta_p(x) \to \beta(x)$ as $p \to \infty$.

Remark 2.2. The relations (14) and (15) are respectively derived from (9) and (10), and by a change of variable, (16) is obtained from (15).

Proposition 2.3. *The function $\beta_p(x)$ satifies the functional equation*

$$\beta_p(x+1) = \frac{1}{x} - \frac{1}{x+2(p+1)} - \beta_p(x). \tag{17}$$

Proof. By using representation (16), we obtain

$$\begin{aligned}
\beta_p(x+1) + \beta_p(x) &= \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^x\, dt + \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{x-1}\, dt \\
&= \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^x \left(\frac{t+1}{t}\right) dt \\
&= \int_0^1 (1-t^{2(p+1)}) t^{x-1}\, dt \\
&= \frac{1}{x} - \frac{1}{x+2(p+1)},
\end{aligned}$$

which completes the proof. □

As an immediate consequence of (17), we obtain the upper bound

$$\beta_p(x) \le \frac{1}{x} - \frac{1}{x+2(p+1)}. \tag{18}$$

Also, successive applications of (17) yields the generalized form

$$\beta_p(x+n) = \sum_{s=0}^{n-1} \frac{(-1)^{s+n+1}}{x+s} + \sum_{s=0}^{n-1} \frac{(-1)^{s+n}}{x+s+2(p+1)} + (-1)^n \beta_p(x),\ n \in \mathbb{N}. \quad (19)$$

Also, successive differentiations of (13), (15), (16) and (17) yields respectively

$$\beta_p^{(n)}(x) = \frac{1}{2^{n+1}} \left\{ \psi_p^{(n)}\left(\frac{x+1}{2}\right) - \psi_p^{(n)}\left(\frac{x}{2}\right) \right\}, \quad (20)$$

$$= (-1)^n \int_0^\infty \frac{t^n - t^n e^{-2(p+1)t}}{1+e^{-t}} e^{-xt}\, dt, \quad (21)$$

$$= \int_0^1 \frac{(\ln t)^n - (\ln t)^n t^{2(p+1)}}{1+t} t^{x-1}\, dt, \quad (22)$$

$$\beta_p^{(n)}(x+1) = \frac{(-1)^n n!}{x^{n+1}} - \frac{(-1)^n n!}{(x+2(p+1))^{n+1}} - \beta_p^{(n)}(x), \quad (23)$$

where $n \in \mathbb{N}_0$ and $\beta_p^{(n)}(x) \to \beta^{(n)}(x)$ as $p \to \infty$.

Remark 2.4. It follows easily from (20)-(22) that:

(a) $\beta_p(x)$ is positive and decreasing,

(b) $\beta_p^{(n)}(x)$ is positive and decreasing if $n \in \mathbb{N}_0$ is even,

(c) $\beta_p^{(n)}(x)$ is negative and increasing if $n \in \mathbb{N}_0$ is odd.

Theorem 2.5. *The function* $\beta_p(x)$ *satisfies the inequality*

$$\beta_p\left(\frac{x}{u} + \frac{y}{v}\right) \leq [\beta_p(x)]^{\frac{1}{u}} [\beta_p(y)]^{\frac{1}{v}}, \quad x, y \in (0,\infty), \quad (24)$$

where $u > 1$, $v > 1$ *and* $\frac{1}{u} + \frac{1}{v} = 1$. *Put in another way, the function* $\beta_p(x)$ *is logarithmically convex on* $(0,\infty)$.

Proof. Let $u > 1$, $v > 1$ and $\frac{1}{u} + \frac{1}{v} = 1$ and $x, y \in (0,\infty)$. Then Hölder's inequality implies

$$\begin{aligned}
\beta_p\left(\frac{x}{u} + \frac{y}{v}\right) &= \int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{\frac{x}{u}+\frac{y}{v}-1}\, dt \\
&= \int_0^1 \left(\frac{1-t^{2(p+1)}}{1+t} t^{x-1}\right)^{\frac{1}{u}} \left(\frac{1-t^{2(p+1)}}{1+t} t^{y-1}\right)^{\frac{1}{v}} dt \\
&\leq \left(\int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{x-1}\, dt\right)^{\frac{1}{u}} \left(\int_0^1 \frac{1-t^{2(p+1)}}{1+t} t^{y-1}\, dt\right)^{\frac{1}{v}} \\
&= [\beta_p(x)]^{\frac{1}{u}} [\beta_p(y)]^{\frac{1}{v}},
\end{aligned}$$

which gives the desired result. □

Remark 2.6. As a by-product of Theorem 2.5, we obtain immediately the following results.

(a) The inequality $\beta_p(x)\beta_p''(x) \geq \left(\beta_p'(x)\right)^2$ holds for $x \in (0, \infty)$.

(b) The function $\frac{\beta_p'(x)}{\beta_p(x)}$ is increasing on $(0, \infty)$.

Corollary 2.7. *The inequalities*

$$[\beta_p(x+y)]^2 < \beta_p(x)\beta_p(y), \tag{25}$$

$$\beta_p(x+y) < \beta_p(x) + \beta_p(y), \tag{26}$$

hold for $x, y \in (0, \infty)$.

Proof. Let $u = v = 2$ in Theorem 2.5. Then by the decreasing property of $\beta_p(x)$, it follows easily that

$$\beta_p(x+y) < \beta_p\left(\frac{x+y}{2}\right) \leq \sqrt{\beta_p(x)\beta_p(y)}, \tag{27}$$

which gives (25). Next, by (27) and the basic AM-GM inequality, we obtain

$$\beta_p(x+y) < \sqrt{\beta_p(x)\beta_p(y)} \leq \frac{\beta_p(x)}{2} + \frac{\beta_p(y)}{2} \leq \beta_p(x) + \beta_p(y),$$

which gives (26). □

Corollary 2.8. *The inequality*

$$1 < \frac{\beta_p(z)}{\beta_p(z+1)} < \frac{\beta_p(z-1)}{\beta_p(z)} \tag{28}$$

holds for $z > 1$.

Proof. Let $z > 1$. Then the left-hand side of (28) follows directly from the decreasing property of $\beta_p(x)$. Next, by letting $x = z - 1$ and $y = z + 1$ in right-hand side of (27), we obtain

$$\beta_p^2(z) < \beta_p(z-1)\beta_p(z+1), \tag{29}$$

which when rearranged, gives the right-hand side of (28). Alternatively, we could proceed as follows. Let $f(x) = \frac{\beta_p(x)}{\beta_p(x+1)}$ for $x > 0$. Then

$$f'(x) = f(x)\left[\frac{\beta_p'(x)}{\beta_p(x)} - \frac{\beta_p'(x+1)}{\beta_p(x+1)}\right] < 0,$$

which implies that $f(x)$ is decreasing. Hence $f(z) < f(z-1)$ which also gives the right-hand side of (28). □

Theorem 2.9. *The function*

$$\phi(x) = u^x \beta_p(x), \quad u > 0, \tag{30}$$

is convex on $(0, \infty)$.

Proof. Let $a > 1$, $b > 1$, $\frac{1}{a} + \frac{1}{b} = 1$ and $x, y \in (0, \infty)$. Then the log-convexity of $\beta_p(x)$ implies

$$\phi\left(\frac{x}{a} + \frac{y}{b}\right) = u^{\frac{x}{a}+\frac{y}{b}} \beta_p\left(\frac{x}{a} + \frac{y}{b}\right) \leq \left[u^x \beta_p(x)\right]^{\frac{1}{a}} \left[u^y \beta_p(y)\right]^{\frac{1}{b}},$$

and by the classical Young's inequality, we obtain

$$\left[u^x \beta_p(x)\right]^{\frac{1}{a}} \left[u^y \beta_p(y)\right]^{\frac{1}{b}} \leq \frac{u^x \beta_p(x)}{a} + \frac{u^y \beta_p(y)}{b} = \frac{\phi(x)}{a} + \frac{\phi(y)}{b}.$$

Hence, $\phi(x)$ is convex on $(0, \infty)$. □

Theorem 2.10. *The inequality*

$$\exp\left\{\beta_p\left(x + \frac{1}{2}\right)\right\} \leq \frac{\Gamma_p\left(\frac{x}{2}+1\right)\Gamma_p\left(\frac{x}{2}\right)}{\Gamma_p^2\left(\frac{x}{2}+\frac{1}{2}\right)} \leq \exp\left\{\frac{1}{2x} - \frac{1}{2x + 4(p+1)}\right\} \tag{31}$$

holds for $x > 0$.

Proof. We make use of the Hermite-Hadamard's inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(s)\,ds \leq \frac{f(a) + f(b)}{2}, \tag{32}$$

for a convex function $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$. Since every logarithmically convex function is also convex, it follows that $\beta_p(x)$ is convex. Now, letting $f(s) = \beta_p(s) = \frac{1}{2}\left\{\psi_p\left(\frac{s+1}{2}\right) - \psi_p\left(\frac{s}{2}\right)\right\}$, $a = x > 0$ and $b = x + 1$ in (32) gives

$$\beta_p\left(x + \frac{1}{2}\right) \leq \left|\ln\Gamma_p\left(\frac{x}{2} + \frac{1}{2}\right) - \ln\Gamma_p\left(\frac{x}{2}\right)\right|_x^{x+1} \leq \frac{\beta_p(x+1) + \beta_p(x)}{2},$$

which by (17) implies

$$\beta_p\left(x + \frac{1}{2}\right) \leq \ln\frac{\Gamma_p\left(\frac{x}{2}+1\right)\Gamma_p\left(\frac{x}{2}\right)}{\Gamma_p^2\left(\frac{x}{2}+\frac{1}{2}\right)} \leq \frac{1}{2}\left(\frac{1}{x} - \frac{1}{x + 2(p+1)}\right).$$

Then by exponentiation, we obtain the required result (31). □

Remark 2.11. The function $\frac{\Gamma_p\left(\frac{x}{2}+1\right)\Gamma_p\left(\frac{x}{2}\right)}{\Gamma_p\left(\frac{x}{2}+\frac{1}{2}\right)}$ is a special case of

$$T_p(x,y) = \frac{\Gamma_p(x)\Gamma_p(y)}{\Gamma_p^2(\frac{x+y}{2})}, \quad x, y > 0,$$

which is a p-analogue of Gurland's ratio [4] for the Gamma function. For more information concerning the Gurland's ratio, one may refer to [7] and the related references therein.

Lemma 2.12. *Let $f(t)$ and $g(t)$ be any two functions with convolution $f * g = \int_0^t f(s)g(t-s)\,ds$. Then the Laplace transform of the convolution is given as*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\,\mathcal{L}\{g\}.$$

That is

$$\int_0^\infty \left[\int_0^t f(s)g(t-s)\,ds\right] e^{-xt}\,dt = \int_0^\infty f(t)e^{-xt}\,dt \int_0^\infty g(t)e^{-xt}\,dt. \tag{33}$$

The above lemma is well-known in the literature as the the convolution theorem for Laplace transforms. We shall rely on it in proving some of the results that follow.

Theorem 2.13. *The function $Q(x) = x\beta_p(x)$ is completely monotonic on $(0,\infty)$.*

Proof. Recall that a function $f : (0,\infty) \to \mathbb{R}$ is said to be completely monotonic on $(0,\infty)$ if f has derivatives of all order and $(-1)^n f^{(n)}(x) \geq 0$ for all $x \in (0,\infty)$ and $n \in \mathbb{N}$. By repeated differentiation, we obtain

$$Q^{(n)}(x) = n\beta_p^{(n-1)}(x) + x\beta_p^{(n)}(x). \tag{34}$$

Then by (12), (15) and (33), we obtain

$$\begin{aligned}
\frac{(-1)^n Q^{(n)}(x)}{x} &= (-1)^n \left[\frac{n}{x}\beta_p^{(n-1)}(x) + \beta_p^{(n)}(x)\right] \\
&= -n\int_0^\infty e^{-xt}\,dt \int_0^\infty \frac{t^{n-1}(1-e^{-2(p+1)t})}{1+e^{-t}} e^{-xt}\,dt \\
&\quad + \int_0^\infty \frac{t^n(1-e^{-2(p+1)t})}{1+e^{-t}} e^{-xt}\,dt \\
&= -n\int_0^\infty \left[\int_0^t \frac{s^{n-1}(1-e^{-2(p+1)s})}{1+e^{-s}}\,ds\right] e^{-xt}\,dt \\
&\quad + \int_0^\infty \frac{t^n(1-e^{-2(p+1)t})}{1+e^{-t}} e^{-xt}\,dt \\
&= \int_0^\infty W(t)e^{-xt}\,dt,
\end{aligned}$$

where

$$W(t) = -n\int_0^t \frac{s^{n-1}(1-e^{-2(p+1)s})}{1+e^{-s}}\,ds + \frac{t^n(1-e^{-2(p+1)t})}{1+e^{-t}}.$$

Then $W(0) = \lim_{t\to 0} W(t) = 0$. In addition,

$$W'(t) = 2(p+1)\frac{t^n e^{-2(p+1)t}}{1+e^{-t}} + t^n e^{-t}\frac{1-e^{-2(p+1)t}}{(1+e^{-t})^2} > 0,$$

which implies that $W(t)$ increasing. Hence for $t > 0$, we have $W(t) > W(0) = 0$. Therefore,

$$(-1)^n Q^{(n)}(x) \geq 0 \tag{35}$$

which concludes the proof. □

Remark 2.14. Theorem 2.13 implies that $Q(x) = x\beta_p(x)$ is decreasing and convex. These further imply that

$$\beta_p(x) + x\beta_p'(x) < 0 \tag{36}$$

and

$$2\beta_p'(x) + x\beta_p''(x) > 0 \tag{37}$$

respectively.

Corollary 2.15. *The function $H(x) = x\beta_p'(x)$ is increasing and concave on $(0,\infty)$.*

Proof. By (34), (35) and (37), we obtain

$$H'(x) = \beta_p'(x) + x\beta_p''(x) > 2\beta_p'(x) + x\beta_p''(x) > 0,$$

$$H''(x) = 2\beta_p''(x) + x\beta_p'''(x) < 3\beta_p''(x) + x\beta_p'''(x) < 0,$$

which conclude the proof. □

Theorem 2.16. *The inequality*

$$\beta_p(xy) \leq \beta_p(x) + \beta_p(y), \tag{38}$$

holds for $x > 0$ and $y \geq 1$.

Proof. Let $\phi(x,y) = \beta_p(xy) - \beta_p(x) - \beta_p(y)$ for $x > 0$ and $y \geq 1$. By fixing y, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}\phi(x,y) &= y\beta_p'(xy) - \beta_p'(x)\\ &= \frac{1}{x}\left[xy\beta_p'(xy) - x\beta_p'(x)\right]\\ &\geq 0,\end{aligned}$$

since $x\beta_p'(x)$ is increasing. Hence, $\phi(x,y)$ is increasing. Then for $0 < x < \infty$, we obtain

$$\phi(x,y) \leq \lim_{x\to\infty} \phi(x,y) = -\beta_p(y) < 0,$$

which gives the result (38). □

Remark 2.17. Note that $\left|\beta_p^{(n)}(x)\right| = (-1)^n \beta_p^{(n)}(x)$ for all $n \in \mathbb{N}_0$. In respect of this, the recurrence relation (23) yields

$$\left|\beta_p^{(n)}(x+1)\right| = \frac{n!}{x^{n+1}} - \frac{n!}{(x+2(p+1))^{n+1}} - \left|\beta_p^{(n)}(x)\right|. \tag{39}$$

It is also worth noting that, if $F(x) = \left|\beta_p^{(n)}(x)\right|$, then $F'(x) = -\left|\beta_p^{(n+1)}(x)\right|$. This implies that the $\left|\beta_p^{(n)}(x)\right|$ is decreasing for all $n \in \mathbb{N}_0$. Furthermore, it follows readily from (39) that

$$\left|\beta_p^{(n)}(x)\right| \leq \frac{n!}{x^{n+1}} - \frac{n!}{(x+2(p+1))^{n+1}}. \tag{40}$$

This is a generalization of (18).

Theorem 2.18. *Let Δ_n be defined for $x > 0$ and $n \in \mathbb{N}_0$ as*

$$\Delta_n(x) = \frac{x^{n+1}}{n!} \left|\beta_p^{(n)}(x)\right|. \tag{41}$$

Then,

(a) $\lim_{x\to 0} \Delta_n(x) = 1$ *and* $\lim_{x\to 0} \Delta_n'(x) = 0$.

(b) $\Delta_n(x)$ *is decreasing.*

Proof. (a) By virtue of (39), we obtain

$$\lim_{x\to 0} \Delta_n(x) = \lim_{x\to 0} \left\{ 1 - \left(\frac{x}{x+2(p+1)}\right)^{n+1} - \frac{x^{n+1}}{n!}\left|\beta_p^{(n)}(x+1)\right| \right\} = 1.$$

Also,

$$\begin{aligned}
\lim_{x\to 0} \Delta_n'(x) &= \lim_{x\to 0} \left\{ \frac{(n+1)x^n}{n!}\left|\beta_p^{(n)}(x)\right| - \frac{x^{n+1}}{n!}\left|\beta_p^{(n+1)}(x)\right| \right\} \\
&= \lim_{x\to 0} \left\{ \frac{(n+1)x^{n+1}}{(x+2(p+1))^{n+2}} - \frac{(n+1)x^n}{(x+2(p+1))^{n+1}} \right. \\
&\quad \left. + \frac{x^{n+1}}{n!}\left|\beta_p^{(n+1)}(x+1)\right| - (n+1)\frac{x^{n+1}}{n!}\left|\beta_p^{(n)}(x+1)\right| \right\} \\
&= 0.
\end{aligned}$$

(b) By using (21) and (33), we obtain

$$\begin{aligned}
\frac{n!}{x^{n+1}}\Delta_n'(x) &= \frac{n+1}{x}\left|\beta_p^{(n)}(x)\right| - \left|\beta_p^{(n+1)}(x)\right| \\
&= (n+1)\int_0^\infty e^{-xt}\,dt\int_0^\infty \frac{t^n(1-e^{-2(p+1)t})}{1+e^{-t}}e^{-xt}\,dt \\
&\quad - \int_0^\infty \frac{t^{n+1}(1-e^{-2(p+1)t})}{1+e^{-t}}e^{-xt}\,dt \\
&= (n+1)\int_0^\infty \left[\int_0^t \frac{s^n(1-e^{-2(p+1)s})}{1+e^{-s}}\,ds\right] e^{-xt}\,dt \\
&\quad - \int_0^\infty \frac{t^{n+1}(1-e^{-2(p+1)t})}{1+e^{-t}}e^{-xt}\,dt \\
&= \int_0^\infty K(t)e^{-xt}\,dt,
\end{aligned}$$

where

$$K(t) = (n+1)\int_0^t \frac{s^n(1-e^{-2(p+1)s})}{1+e^{-s}}\,ds - \frac{t^{n+1}(1-e^{-2(p+1)t})}{1+e^{-t}}.$$

Then $K(0) = \lim_{t\to 0} K(t) = 0$. Furthermore,

$$K'(t) = -2(p+1)\frac{t^{n+1}e^{-2(p+1)t}}{1+e^{-t}} - t^{n+1}e^{-t}\frac{1-e^{-2(p+1)t}}{(1+e^{-t})^2} < 0,$$

which implies that $K(t)$ decreasing. Hence for $t>0$, we have $K(t) < K(0) = 0$. Therefore $\Delta_n'(x) < 0$ which gives the desired result. □

3 Open Problem

The function $x\beta_p(x)$ has been shown to be completely monotonic in Theorem 2.13. Show that the generalized form $\frac{x^{n+1}}{n!}\left|\beta_p^{(n)}(x)\right|$, $x>0$, $n\in\mathbb{N}_0$ is completely monotonic.

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