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# Fuglede-Putnam theorem for $p$-$w$-hyponormal or class $\mathcal{Y}$ operators

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**Abstract**

***An asymmetric Fuglede-Putnam Theorem for*** $p$-$w$***-phyponormal operators and class*** $\mathcal{Y}$ ***operators is proved. As a consequence of this result, we obtain that the range of generalized derivation induced by these classes of operators is orthogonal to its kernel.***

**Keywords:** *Fuglede-Putnam theorem, p- w-hyponormal operator, class* $\mathcal{Y}$ *operator.*
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## 1 Introduction

For complex spaces $\mathcal{H}$ and $\mathcal{K}$, let $\mathcal{B}(\mathcal{H})$ $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the algebra of all bounded operators on $\mathcal{H}$ , the algebra of all bounded operators on $\mathcal{K}$ and the set of all bounded transformations from $\mathcal{H}$ to $\mathcal{K}$ respectively.

A bounded operator $A \in \mathcal{B}(\mathcal{H})$, set , as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A^*|^2 - |A|^2$ ( the self commutator of $A$) and consider the following definitions: $A$ is normal if $A^*A = AA^*$, hyponormal if $A^*A \geq AA^*$, $p$-hyponormal if $|A|^{2p} \geq |A^*|^{2p}$ for $0 < p < 1$ and semi hyponormal if $|A| \geq |A^*|$ . The lowner-Heinz inequality implies that if $A$ is $q$-hyponormal, then $A$ is $p$-hyponormal for all $0 < p \leq q$. An invertible operator $A \in \mathcal{B}(\mathcal{H})$ is called *log*-hyponormal if $\log(A^*A) = \log(AA^*)$. Clearly every invertible $p$-hyponormal operator is log hyponormal.

Let $A = U|A|$ be the polar decomposition of $A$. A. Aluthge[1] defined the operator $\widetilde{A} = |A|^{\frac{1}{2}}U|A|^{\frac{1}{2}}$ which is called the Aluthge transformation of $A$. An operator $A$ is said to be $w$-hyponormal if $|\widetilde{A}| \geq |A| \geq |\widetilde{A}^*|$ . An operator $A$ is said to be $p$-$w$ - hyponormal if $|\widetilde{A}|^p \geq |A|^p \geq |\widetilde{A}^*|^p$ [13]. It is well known that the class of $w$ - hyponormal operators contains, both $p$- and log - hyponormal operators [2]. These classes are related by proper inclusion

$$\text{hyponormal} \subset p\text{-hyponormal} \subset w\text{-hyponormal} \subset p\text{-}w\text{-hyponormal}.$$

It is well known that if $A$ is $w$-hyponormal, then $\widetilde{A}$ is semi- hyponoormal and if $A$ is $p$-$w$-hyponormal, then $\widetilde{A}$ is $\frac{p}{2}$- hyponormal [13].

An operator $A$ is said to be class $\mathcal{Y}_\alpha$ for $\alpha \geq 1$ if there exist a positive number $k_\alpha$ such that

$$|AA^* - A^*A|^\alpha \leq k_\alpha^2 (A - \lambda)^*(A - \lambda) \text{ for all } \lambda \in \mathbb{C}.$$

If $1 \leq \alpha \leq \beta$, then $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$. Recall that an operator $A \in \mathcal{B}(\mathcal{H})$ is said to be dominant if for each $\lambda \in \mathbb{C}$ there exists a positive number $M_\lambda$ such that

$$(A - \lambda)(A - \lambda)^* \leq M_\lambda (A - \lambda)^*(A - \lambda).$$

If the constants $M_\lambda$ are bounded by a positive operator $M$, then $A$ is said to be $M$-hyponormal. Evidently $M$-hyponormal operators are dominant. Let $\mathcal{Y} = \bigcup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that $M$-hyponormal are class $\mathcal{Y}_2$ [11].

The famous Fuglede-Putnam theorem ( see., [7, 9]) asserts that if $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are normal and $AX = BX$ for some $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*X = XB^*$. Fuglede-Putnam's Theorem for $p$-hyponormal operators and $\mathcal{Y}$ operators was studied by Mecheri et. al [8], Fuglede-Putnam's Theorem for $w$-hyponormal operators and $\mathcal{Y}$ operators was extensively studied in [4].

Let $A, B \in \mathcal{H}$, we define the generalized derivation $\delta_{A,B}$ induced by $A$ and $B$ as follows

$$\delta_{A,B}(X) = AX - XB \text{ for all } X \in \mathcal{B}(\mathcal{H})$$

J. Anderson and C. Foias [3] proved that if $A$ and $B$ are normal, $S$ is an operator such that $AS = SB$, then

$$||\delta_{A,B}(X) - S|| \geq ||S|| \text{ for all } X \in \mathcal{B}(\mathcal{H})$$

where $||.||$ is the usual operator norm. Hence the range of $\delta_{AB}$ is orthogonal to the null space of $\delta_{AB}$. The orthogonality here is understood to be in the sense of Birkhoff-James [3].

The purpose of this paper is to prove that the Fuglede - Putnam theorems remains true if $A \in \mathcal{B}(\mathcal{H})$ is $p$-$w$ - hyponormal and $B^* \in \mathcal{B}(\mathcal{K})$ is class $\mathcal{Y}$ operator. As a consequence of this result, we give a similar orthogonality result by proving that the range of the generalized derivation induced by above classes of operators is orthogonal to its kernel.

# 2 Preliminaries

The following result was proved in [13] by Yang Changsen Li Haiying.

**Lemma 2.1** *[13] Let $A$ be p-w-hyponormal operator. If $\widetilde{A}$ is normal, then $A = \widetilde{A}$.*

The following theorem due to Duggal[6] is well known and useful.

**Theorem 2.2** *[6] If $A, B^*$ are p-hyponormal operators satisfying $AX = XB$ for some operator $X$, then $A^*X = XB^*$. $\overline{ranX}$ reduces $A$, $ker^{\perp}X$ reduces $B$ and $A|_{\overline{ranX}}$, $B|_{ker^{\perp}X}$ are unitarily equivalent normal operators.*

**Theorem 2.3** *[10] Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$. Then the following assertions are equivalent.*
*(i) $A, B$ satisfy Fuglede - Putnam theorem.*
*(ii) If $AC = CB$ for some operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $\overline{ranC}$ reduces $A$, $(kerC)^{\perp}$ reduces $B$ and $A|_{\overline{ranC}}$, $B|_{(kerC)^{\perp}}$ are normal.*

**Lemma 2.4** *([11]) Let $A \in \mathcal{B}(\mathcal{H})$ be a class $\mathcal{Y}$ operator and $\mathcal{M} \subset \mathcal{H}$ invariant subspace under $A$. If $A|_{\mathcal{M}}$ is normal, then $\mathcal{M}$ reduces $A$.*

Recall that $A \in \mathcal{B}(\mathcal{H})$ is said to have the single valued extension property at $\lambda$ (SVEP for short) if for every neighborhood $\mathcal{U}$ of $\lambda$, the only analytic function $f : \mathcal{U} \to \mathcal{H}$ which satisfies the equation $(A - \lambda)f(\lambda) = 0$ for all $\lambda \in \mathcal{U}$ is the function $f = 0$. We say that $A \in \mathcal{B}(\mathcal{H})$ satisfies the SVEP property if $A$ has the single valued extension property at every $\lambda \in \mathbb{C}$. It is well known that the class of $p$-hyponormal operators satisfies SVEP.

# 3 Main Results

**Lemma 3.1** *Let $A$ be p-w-hyponormal operator and $\mathcal{M} \subset \mathcal{H}$ be an invariant subspace of $A$. Then the restriction $A \mid \mathcal{M}$ is p-w-hyponormal.*

**Proof.**
Let $P$ be the orthogonal projection on $\mathcal{M}$. From $AP = PAP$, we deduce the following inequality by using Lowner-Heinz theorem

$$\mid (AP)^* \mid^p \leq \mid A^* \mid^p \tag{1}$$

Also we deduce,

$$\mid AP \mid \geq P \mid A \mid P \tag{2}$$

and so $P \mid AP \mid P \geq P \mid A \mid P$. Thus we have, $\mid AP \mid \geq \mid A \mid$ holds on $\overline{ran\, P}$. Since $A$ is $p$-$w$-hyponormal operator, we have

$$\mid A^* \mid^p \leq (\mid A^* \mid^{\frac{1}{2}} \mid A \mid\mid A^* \mid^{\frac{1}{2}})^{\frac{p}{2}}$$

Then by (1), we have

$$\mid (AP)^* \mid^p \leq (\mid (AP)^* \mid^{\frac{1}{2}} \mid A \mid\mid (AP)^* \mid^{\frac{1}{2}})^{\frac{p}{2}} \tag{3}$$

Now we have the following inequality

$$\mid (AP)^* \mid^{\frac{1}{2}} \mid A \mid\mid (AP)^* \mid^{\frac{1}{2}} \leq \mid (AP)^* \mid^{\frac{1}{2}} \mid AP \mid\mid (AP)^* \mid^{\frac{1}{2}} . \tag{4}$$

Then by Lowner-Heinz theorem

$$(\mid (AP)^* \mid^{\frac{1}{2}} \mid A \mid\mid (AP)^* \mid^{\frac{1}{2}})^{\frac{p}{2}} \leq (\mid (AP)^* \mid^{\frac{1}{2}} \mid AP \mid\mid (AP)^* \mid^{\frac{1}{2}})^{\frac{p}{2}}. \tag{5}$$

From from (3) and (4), we deduce that

$$\mid (AP)^* \mid^p \leq (\mid (AP)^* \mid^{\frac{1}{2}} \mid AP \mid\mid (AP)^* \mid^{\frac{1}{2}})^{\frac{p}{2}}$$

holds on $\overline{ran\, P}$ and so $AP$ is $p$-$w$-hyponormal. □

**Theorem 3.2** *Let $A \in \mathcal{B}(\mathcal{H})$ be an injective p-w-hyponormal $(0 < p \leq 1)$ and $B^* \in \mathcal{B}(\mathcal{K})$ be class $\mathcal{Y}$. If $AC$ $=CB$ for some operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*C$ $=CB^*$.*

**Proof.** Since $B^* \in$ class $\mathcal{Y}$, there exist a positive $k_\alpha$ for $\alpha \geq 1$ such that

$$|BB^* - B^*B|^\alpha \leq k_\alpha^2 (B - \lambda)^*(B - \lambda) \text{ for all } \lambda \in \mathbb{C}$$

Then by [5], for $x \in |BB^* - B^*B|^{\frac{\alpha}{2}}\mathcal{K}$ there exist a bounded function $f : \mathbb{C} \to \mathcal{K}$ such that

$$(B - \lambda)f(\lambda) = x \text{ for all } \lambda \in \mathbb{C}.$$

Since $A$ is $p$-$w$-hyponormal, then $\widetilde{A}$ is $p$-hyponormal, this yields

$$(\widetilde{A} - \lambda)|A|^{\frac{1}{2}}Cf(\lambda) = |A|^{\frac{1}{2}}(A - \lambda)Cf(\lambda).$$

From $AC = CB$, it follows that

$$\begin{aligned}(\widetilde{A} - \lambda)|A|^{\frac{1}{2}}f(\lambda) &= |A|^{\frac{1}{2}}C(B - \lambda)f(\lambda) \\ &= |A|^{\frac{1}{2}}Cx, \text{ for all } \lambda \in \mathbb{C}.\end{aligned}$$

Now we claim that $|A|^{\frac{1}{2}}Cx = 0$. If $|A|^{\frac{1}{2}}Cx \neq 0$, there exist an entire analytic function $g : \mathbb{C} \to \mathcal{H}$ such that $(A - \lambda)g(\lambda) = |A|^{\frac{1}{2}}Cx$ because $p$-hyponormal has SVEP .

Since $g(\lambda) = (A - \lambda)^{-1}|A|^{\frac{1}{2}}Cx \to 0$ as $n \to \infty$, $g(\lambda) = 0$ by Liouville's theorem. Thus $|A|^{\frac{1}{2}}Cx = 0$. This is a contradiction. Hence

$$|A|^{\frac{1}{2}}C|BB^* - B^*B|^{\frac{\alpha}{2}}\mathcal{K} = 0.$$

Since ker $A$ =ker $|A| = \{0\}$ we get

$$C(BB^* - B^*B) = 0$$

Because $AC = CB$, $\overline{\text{ran}C}$ and $(\ker^{\perp} C)$ are invariant subspaces of $A$ and $B^*$ respectively, we can write $A, B$ and $C$ as

$$A = \begin{pmatrix} A_1 & T \\ 0 & A_2 \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \overline{\text{ran}C} \oplus \overline{\text{ran}C}^{\perp}$$

$$B = \begin{pmatrix} B_1 & 0 \\ S & B_2 \end{pmatrix} \quad \text{on } \ker C^{\perp} \oplus \ker C$$

and

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } \ker C^{\perp} \oplus \ker C \to \overline{\text{ran}C} \oplus \overline{\text{ran}C}^{\perp}$$

and so

$$\begin{aligned} BB^* - B^*B &= \begin{pmatrix} B_1 & 0 \\ S & B_2 \end{pmatrix}\begin{pmatrix} B_1^* & S^* \\ 0 & B_2^* \end{pmatrix}\begin{pmatrix} B_1^* & S^* \\ 0 & B_2^* \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ S & B_2 \end{pmatrix} \\ &= \begin{pmatrix} B_1B_1^* - B_1^*B_1 - S^*S & B_1S^* - S^*B_1 \\ (B_1S^* - S^*B_2)^* & SS^* + B_2B_2^* - B_2^*B_2 \end{pmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} 0 &= C(BB^* - B^*B) \\ &= \begin{pmatrix} C_1(B_1B_1^* - B_1^*B_1 - S^*S) & C_1(B_1S^* - S^*B_1) \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since $C_1$ is injective and has dense range,

$$B_1B_1^* - B_1^*B_1 - S^*S = 0.$$

Hence,

$$B_1B_1^* = B_1^*B_1 + S^*S$$

Therefore $B_1^*$ is hyponormal.

Since $AC = CB$, we get $A_1C_1 = C_1B_1$ where $A_1$ is $p - w$ hyponormal by Lemma 3.1 and so

$$|A_1|^{\frac{1}{2}}A_1C_1 = |A_1|^{\frac{1}{2}}C_1B_1$$

Since $\widetilde{A_1}$ is $\frac{p}{2}$- hyponormal and $B_1^*$ is hyponormal, applying Theorem 2.3

$$\widetilde{A_1}(|A_1|^{\frac{1}{2}}C_1) = (|A_1|^{\frac{1}{2}}C_1)B_1^*.$$

Applying Theorem 3.2, it follows that $\widetilde{A_1}|_{ran|A_1|^{\frac{1}{2}}C_1}$ and $B_1|_{ker(|A_1|^{\frac{1}{2}}C_1)}$ are normal.

Since $|A_1|^{\frac{1}{2}}C_1$ is injective, we have $[ker(|A_1|^{\frac{1}{2}}C_1)]^{\perp} = (kerC)^{\perp}$ and $\overline{ran|A_1|^{\frac{1}{2}}C_1}$ = ran $C$ and hence $\widetilde{A_1}$ is normal. Therefore $A_1$ is normal by Lemma 2.1. Hence

$$A^*C = \begin{pmatrix} A_1^*C_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} C_1B_1^* & 0 \\ 0 & 0 \end{pmatrix} = CB^*.$$

□

**Theorem 3.3** *If $A \in \mathcal{B}(\mathcal{H})$ be p-w hyponormal $(0 < p \leq 1)$ such that $\ker A \subset \ker A^*$ and $B^* \in \mathcal{B}(\mathcal{K})$ be class $\mathcal{Y}$. If $AC = CB$ for some operator $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, then $A^*C = CB^*$.*

**Proof.** Decompose $A$ into normal part $A_1$ and pure part $A_2$ as

$$A = A_1 \oplus A_2 \quad \text{on} \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$$

and let

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \text{on} \quad \mathcal{K} \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

Since $\ker A_2 \subset \ker A_2^*$ and $A_2$ is pure, $A_2$ is injective. $AC = CB$ implies

$$\begin{pmatrix} A_1C_1 \\ A_2C_2 \end{pmatrix} = \begin{pmatrix} C_1B_1 \\ C_2B_2 \end{pmatrix}$$

Hence

$$A^*C = \begin{pmatrix} A_1^*C_1 \\ A_2^*C_2 \end{pmatrix} = \begin{pmatrix} C_1B_1^* \\ C_2B_2^* \end{pmatrix} = CB^*$$

by Theorem 3.2. □

**Theorem 3.4** *If $A \in \mathcal{B}(\mathcal{H})$ be $p - w$ - hyponormal $(0 < p \leq 1)$ such that $\ker A \subset \ker A^*$ and $B^* \in \mathcal{B}(\mathcal{K})$ be class $\mathcal{Y}$, then the range of $\delta_{AB}$ is orthogonal to the null space of $\delta_{AB}$.*

**Proof.** The pair $(A, B)$ verify the Fuglede-Putnam theorem by Theorem 3.2. Let $C \in \mathcal{B}(\mathcal{H})$. According to the following decompositions of $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}_1 = \overline{\text{ran}C} \oplus \overline{\text{ran}C}^{\perp}, \mathcal{H} = \mathcal{H}_2 = \ker C^{\perp} \oplus \ker C$$

we can write $A, B, C$ and $X$

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, C = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}, C = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix},$$

where $A_1$ and $B_1$ are normal operators and $X$ is an operator from $H_1$ to $H_2$. Since $AC = CB$, $A_1C_1 = C_1B_1$. Hence

$$AX - XB - C = \begin{pmatrix} A_1X_1 - X_1B_1 - C_1 & A_2X_2 - X_2B_2 \\ A_1X_3 - X_3B_1 & A_2X_4 - X_4B_2 \end{pmatrix}$$

Since $C_1 \in \ker(\delta_{A_1,B_1})$ and $A_1$ and $B_1$ are normal operators , it yields

$$||AX - XB - C|| \geq ||A_1X_1 - X_1B_1 - C_1|| \geq ||C_1|| = ||C||, \text{ for all } X \in B(H)$$

This means that the range of $\delta_{A,B}$ is orthogonal to the null space of $\delta_{A,B}$. □

# 4 Open Problem

The open problem here is to find classes of nonnormal of operators satisfying the Fuglede-Putnam Property and consequently we obtain the range kernel orthogonality results.

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